SOLUTION OF CUBIC & QUARTIC EQUATIONS

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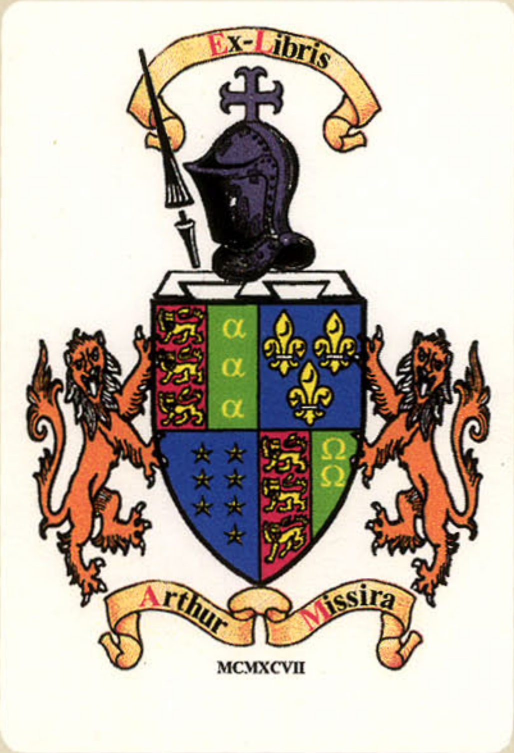

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BY

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PREFACE

This book suggests a rapid and efficient method of computing the roots of an arbitrary cubic equation with real coefficients, by using specially computed 5-figure tables, and also a method of factorizing an arbitrary quartic equation by an appropriate use of a resolvent cubic. A few examples illustrate both methods.

Acknowledgement is due to Mrs. G. Helyer who computed the tables.

Errata

The formulae (2.3) and (2.6), on pages 5 and 6, should read:

$$x = \frac{-b - 2y\sqrt{(b^2 - 3ac)}}{3a} \qquad (2.3)$$

$$x = \frac{-b - 2z\sqrt{(3ac - b^2)}}{3a} \qquad (2.6)$$

CHAPTER 1

INTRODUCTION

THE cubic equation was first solved in an elementary way in the sixteenth century by S. de Ferro, N. Fontano (Tartaglia) and G. Cardano$^{(1)}$, who determined the roots in terms of coefficients by using square and cube roots. The quartic equation was factorized into two quadratics first by Ferrari who reduced the problem to the solution of an auxiliary ("resolvent") cubic equation, as given also by Cardano$^{(1)}$. Since that time, many improvements and alternatives have been introduced$^{(2,3,4,5)}$ such as the solution of cubic equation by hyperbolic and trigonometric functions (the latter especially for Cardano's *casus irreducibilis* of three real roots), also alternative cubic resolvents for quartic equation due to Descartes, Euler and Lagrange. It is, of course, impossible to add anything now to the basic theory of these equations. However, there is still scope to improve the computational technique as, until now, the actual procedure of solution has been rather long and clumsy. That this is so may be inferred from the fact that, in practice, the rigorous algebraic solution was often replaced by various approximate or iterative methods for particular types of cubics and quartics$^{(6,7,8,9)}$. These methods were considered more rapid than the classical ones. However, there exist no such methods which would be convenient for *every* cubic or quartic irrespective of the values of coefficients, and, in many cases, no existing approximate formulae are valid, and all iterative procedures become divergent or very long. In such cases, the classical methods are still most reliable, and it seems desirable to bring them to the highest possible degree of efficiency, which is the purpose of the present book. This seems worth while, as cubic and quartic equations appear very frequently in all fields of applied mathematics, in particular in problems of stability.

It may be argued that the modern digital computers render all algebraic methods obsolete. This is not so, however, for the following reasons.

1. Digital computers are not always available and, even if they are, they should be better used for less elementary problems.
2. The classical methods may be brought to such a degree of efficiency as to work more rapidly than digital computers if the time of programming the latter is taken into account. This is true unless a great number of equations are to be solved.
3. The algebraical methods give a wide possibility of discussion in a variety of practical problems and also of working out new useful approximations for many special cases.
4. The present need for algebraic solutions is also proved by the fact that there appear numerous papers proposing various ways of solving cubic and quartic equations appearing in some special physical problems (to quote only References 15, 16 and 17 dealing with aeronautical problems). All this work would be superfluous if a simple general method were available.

The general cubic equation (2.1) is dealt with in section 2.1. The basic observation is that it may be brought to one of the two standard equations (2.4) or (2.7), with only one variable parameter Δ. The roots of these standard cubics can be easily tabulated against Δ and then the solution of every cubic equation is given by formulae (2.3) or (2.6), analogous to the usual formulae for quadratic equations. Tables 1 and 2 have been computed for this purpose, with a view to current needs, when not a very high degree of accuracy is required. If the method is found convenient, it will perhaps be useful to work out more extensive tables, with a greater number of decimals. The general method fails only in the case $b^2-3ac=0$, but then an even simpler solution exists. This, and several other cases offering simpler solutions, are considered in section 2.2.

There exist already several tables of roots of cubic equations. There is no need to mention some old tables, of little value and generally inaccessible. It suffices to quote more recent tables by Katz(11), Nogrady(12), Zavrotsky(13) and Emde(14). None of them is convenient or sufficient for an average user. The table of Katz(11) is very small and gives only three or even two decimals. Nogrady(12) deals with a cubic in the form $z^3+nz+n=0$ (with a single parameter n) but, instead of tabulating three roots against n, tabulates n against a real root z_1 (the latter with three decimals only), so that a long and clumsy procedure is required in each case to improve accuracy and to compute the two remaining roots. Zavrotsky(13) gives an unnecessarily voluminous table (with five decimals) of the equation $x^3+px+q=0$, with *two parameters* p, q, and the value of this table is small as it requires *double interpolation* whenever p and q are not integers. The only useful table is that of Emde(14) which gives roots of the equation $y^3+2=3py$, but its value is greatly diminished by the fact that many entries contain only four or even three decimals, and the transformation of a general cubic to the particular form chosen involves extraction of a cube root. The tables presented here, and the transformation of a general cubic given in section 2.1, are believed to reduce the computation in each particular case to the barest minimum.

The *general quartic equation* (3.1) is considered in section 3.1. The factorization into two quadratics through the resolvent cubic of the form (3.9) is suggested as most convenient. When one root of this cubic has been found, the coefficients of factors are given by simple formulae (3.8). It is not claimed that the particular cubic resolvent suggested is unknown, but it appears in no textbooks of algebra, and is certainly not in common use. An almost identical equation was introduced by Duncan(10) for a special purpose but not shown to be usable as a general resolvent.[†] The method proposed here may fail only if one of the roots equals $B^2/4A$, but then an even simpler solution exists. This, and several other cases offering simpler solutions, are dealt with in section 3.2.

Section 4 gives several numerical examples which show the rapidity of the procedures suggested. The problems of references 15, 16 and 17 are among the examples, showing invariably a higher efficiency than the techniques used in those references.

Appendix I explains the computation of Tables 1 and 2, and Appendix II deals with a special case when Δ is near to 1 in equation (2.4), which makes the interpolation of Table 1 inconvenient. Appendix III contains some remarks on alternative resolvent cubics for factorizing quartics.

[†] See also Babister.[(18)]

CHAPTER 2

CUBIC EQUATION

2.1 MAIN FORMULAE AND TABLES

A general cubic equation with real coefficients:

$$ax^3+bx^2+cx+d = 0 \quad (a > 0) \tag{2.1}$$

can be written in the form

$$(3ax+b)^3-3(b^2-3ac)(3ax+b)+(27a^2d+2b^3-9abc) = 0. \tag{2.2}$$

Two basic cases must be considered:

(A) $b^2-3ac > 0$

Putting

$$x = \frac{-b-2y\surd(b^2-3ac)}{3a}, \tag{2.3}$$

equation (2.2) reduces to

$$4y^3-3y = \Delta, \tag{2.4}$$

where

$$\Delta = \frac{27a^2d+2b^3-9abc}{2|b^2-3ac|^{3/2}}. \tag{2.5}$$

Equation (2.4) contains only one variable parameter Δ, and its three roots (y_1, y_2, y_3) can be easily tabulated against Δ. The equation is illustrated by Fig. 1 where only real roots are shown. It suffices to consider $\Delta \geqslant 0$, because changing the sign of Δ simply leads to all roots changing signs. All three roots are real for $0 \leqslant \Delta \leqslant 1$ but, if $\Delta > 1$, only y_1 is real and y_2, y_3 conjugate complex. A double root is obtained only if $\Delta = 1$, in which case $y_1 = 1$, $y_2 = y_3 = -\frac{1}{2}$.

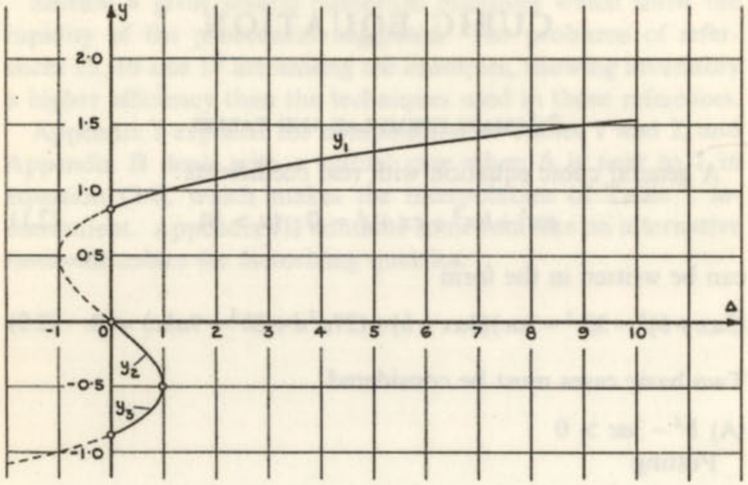

FIG. 1. Graph of equation $4y^3 - 3y = \Delta$.

Table 1 (pp. 39–48) gives the roots (real and complex) of equation (2.4), with 5 decimals, for the following values uf Δ:

0(0·01)1·2(0·1)10(1)170.

First differences are also tabulated. Certain details concerning the table, and some formulae which facilitate the computation of roots, are given in Appendix I.

(B) $b^2 - 3ac < 0$

In this case, we put

$$x = \frac{-b - 2zV(3ac - b^2)}{3a}, \tag{2.6}$$

whereupon equation (2.2) reduces to

$$4z^3 + 3z = \Delta, \tag{2.7}$$

where Δ is still given by (2.5). Equation (2.7) differs essentially from (2.4), and requires a separate tabulation, again only for positive Δ. It is illustrated by Fig. 2. Only one root, z_1, is real, and z_2, z_3 are always conjugate complex. No double root is possible in this case.

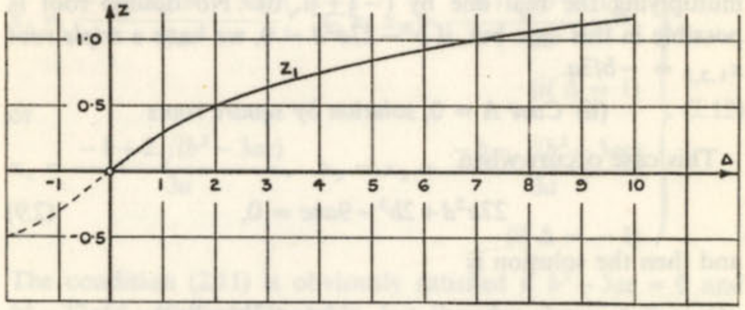

FIG. 2. Graph of equation $4z^3 + 3z = \Delta$.

Table 2 (pp. 49–57), analogous to Table 1, gives roots of (2.7) for the following values of Δ:

0(0·01)1(0·1)10(1)170,

and some details about this table may be found in Appendix I.

The above formulae provide a simple and rapid method for computing the roots. It is only necessary to compute Δ from (2.5), then in case A find y_1, y_2, y_3 from Table 1, and x_1, x_2, x_3 from (2.3); and, similarly, in case B [Table 2, equation (2.6)]. The method fails when $b^2 = 3ac$, but then an even simpler solution exists—see (2.8). This and some other special cases and approximations are considered in section 2.2.

2.2 SPECIAL CASES

(i) *Case $b^2-3ac=0$, solution by cube roots*

The following simple solution is obtained directly from (2.2):

$$x=\frac{-b+\sqrt[3]{(b^3-27a^2d)}}{3a}, \tag{2.8}$$

and obviously only a table of cube roots is needed in this case.† The existing tables give, of course, only real roots, but two complex cube roots may be obtained in each case simply by multiplying the real one by $(-\frac{1}{2}\pm\frac{1}{2}i\sqrt{3})$. No double root is possible in this case but, if $b^3-27a^2d=0$, we have a *triple root* $x_{1,2,3}=-b/3a$.

(ii) *Case $\Delta=0$, solution by square roots*

This case occurs when

$$27a^2d+2b^3-9abc=0, \tag{2.9}$$

and then the solution is

$$x_1=-\frac{b}{3a},\quad x_{2,3}=\frac{-b\pm\sqrt{(3b^2-9ac)}}{3a}, \tag{2.10}$$

so that the three roots form an arithmetical progression.

(iii) *Case of a double root*

We have found already that a double root can exist only when $b^2-3ac>0$, and the condition is then $\Delta=\pm1$. Using (2.5), this condition can be written

$$(27a^2d+2b^3-9abc)^2=4(b^2-3ac)^3 \tag{2.11}$$

or

$$4ac^3+4b^3d+27a^2d^2-18abcd-b^2c^2=0 \tag{2.11a}$$

or, in the deterimental form,

$$\begin{vmatrix} 3 & b & c \\ b & 4ac-b^2 & 3ad \\ c & 3ad & bd \end{vmatrix}=0. \tag{2.11b}$$

If this condition is satisfied, the three roots become

$$\left.\begin{aligned} &x_1=\frac{-b-2\sqrt{(b^2-3ac)}}{3a},\quad x_2=x_3=\frac{-b+\sqrt{(b^2-3ac)}}{3a} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{if } \Delta=1) \\ &\text{or} \\ &x_1=\frac{-b+2\sqrt{(b^2-3ac)}}{3a},\quad x_2=x_3=\frac{-b-\sqrt{(b^2-3ac)}}{3a} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{if } \Delta=-1) \end{aligned}\right\} \tag{2.12}$$

The condition (2.11) is obviously satisfied if $b^2-3ac=0$ and $b^3-27a^2d=0$ simultaneously, but then, as shown before, the equation has a *triple root.*

(iv) *Case of two equal and opposite roots*

This occurs when

$$bc-ad=0. \tag{2.13}$$

Equation (2.1) then factorizes immediately, and the roots are

$$x_1=-\frac{b}{a},\quad x_{2,3}=\pm\sqrt{\left(-\frac{c}{a}\right)}. \tag{2.14}$$

The roots $x_{2,3}$ are both real or both imaginary depending on the sign of c. If (2.1) is a characteristic equation of a linear system of differential equations, and if b, c, d are all positive, then (2.13) provides the known Hurwitz criterion for oscillatory stability: $bc-ad>0$ for the oscillations to be positively damped, and vice versa.

---

† If no such table of sufficient accuracy is available, the following rapidly converging iteration formula for $X=\sqrt[3]{N}$ will be very useful:

$$X_{n+1}=X_n\frac{2+X_n^3/N}{1+2X_n^3/N}.$$

If X_n is in error by a small amount ε, the error in X_{n+1} is approximately $\frac{2}{3}\varepsilon^3$.

(v) *Case of quasi-symmetrical cubic*

Suppose $c = bk$ and $d = ak^3$, so that (2.1) has the form

$$ax^3+bx^2+bkx+ak^3 = 0. \tag{2.15}$$

(This becomes a symmetrical cubic equation if $k = 1$, and an anti-symmetrical one if $k = -1$.)

Equation (2.4) now is

$$4y^3-3y = \frac{3ak+2b}{2}\sqrt{\left(\frac{b-3ak}{b^3}\right)} \quad [b(b-3ak) > 0] \tag{2.16}$$

and has the following roots:

$$y_1 = -\frac{b-3ak}{2\sqrt{[b(b-3ak)]}}, \quad y_{2,3} = \frac{1}{4}\left\{\frac{b-3ak}{\sqrt{[b(b-3ak)]}} \mp 3\sqrt{\left(\frac{b+ak}{b}\right)}\right\}, \tag{2.17}$$

so that the roots of (2.15) are

$$x_1 = -k, \quad x_{2,3} = \frac{ak-b\pm\sqrt{[(b+ak)(b-3ak)]}}{2a}. \tag{2.18}$$

Similarly, (2.7) becomes

$$4z^3+3z = \frac{3ak+2b}{2}\sqrt{\left(\frac{3ak-b}{b^3}\right)} \quad [b(3ak-b) > 0] \tag{2.19}$$

and has the following roots:

$$z_1 = \frac{3ak-b}{2\sqrt{[b(3ak-b)]}}, \quad z_{1,3} = \frac{1}{4}\left\{-\frac{3ak-b}{\sqrt{[b(3ak-b)]}} \mp 3\sqrt{\frac{b+ak}{-b}}\right\}, \tag{2.20}$$

so that the roots of (2.15) are again given by (2.18). These roots can be obtained directly by factorizing (2.15), and the use of tables is superfluous in this case.

(vi) *Case of large* Δ

It is unnecessary to continue tables 1 and 2 for large values of Δ, because the roots of (2.4) or (2.7) can then be determined, with excellent accuracy, by rapidly convergent power series. If we denote

$$p = \sqrt[3]{(\Delta/4)}, \tag{2.21}$$

then (2.4) becomes

$$4y^3-3y = 4p^3, \tag{2.22}$$

and the following expansions for the real and complex roots are easily proved:

$$\left.\begin{aligned} y_1 &= p+\frac{1}{4p}+\frac{0}{p^3}-\frac{1}{192p^5}+\frac{1}{768p^7}+\frac{0}{p^9}\dots,\\ y_{2,3} &= -\tfrac{1}{2}y_1\pm\frac{i\sqrt{3}}{2}\left(p-\frac{1}{4p}+\frac{0}{p^3}-\frac{1}{192p^5}-\frac{1}{768p^7}+\frac{0}{p^9}\dots\right)\end{aligned}\right\} \tag{2.23}$$

Similar expansions apply for the roots of (2.7):

$$\left.\begin{aligned} z_1 &= p-\frac{1}{4p}+\frac{0}{p^3}+\frac{1}{192p^5}+\frac{1}{768p^7}+\frac{0}{p^3}\dots,\\ z_{2,3} &= -\tfrac{1}{2}z_1\pm\frac{i\sqrt{3}}{2}\left(p+\frac{1}{4p}+\frac{0}{p^3}+\frac{1}{192p^5}-\frac{1}{768p^7}+\frac{0}{p^9}\dots\right).\end{aligned}\right\} \tag{2.24}$$

It may be noticed that for $\Delta = 170$ we have $p \approx 3{\cdot}48977$, and the term $1/192p^5$ is equal to $0{\cdot}00001$, very nearly, while the next term is almost 50 times less than this. Therefore, if $\Delta > 170$, it will suffice to take two terms (p and $\frac{1}{4}p$) in each series, and the errors will not exceed one unit in the fifth decimal place. A table of cube roots then suffices to compute the roots very rapidly, and this is why tables 1 and 2 have not been continued beyond $\Delta = 170$.

(vii) *Case of* Δ *very near to* 1 *in* (2.4)

If Δ in equation (2.4) is only slightly less or slightly greater than 1, then the linear interpolation in table 1 gives accurate results for y_1, but not for $y_{2,3}$, and even corrections by second and higher differences may not be very helpful. If a high accuracy is required, it may be convenient to use rapidly convergent power series given in Appendix II.

CHAPTER 3

QUARTIC EQUATION

3.1 FACTORIZATION THROUGH RESOLVENT CUBIC

Let us consider a general quartic equation with real coefficients:

$$f(\lambda) = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \quad (A > 0) \tag{3.1}$$

and suppose that it may be factorized in the form

$$A \, . \, f(\lambda) = (A\lambda^2 + G\lambda + H)(A\lambda^2 + g\lambda + h) = 0, \tag{3.2}$$

where coefficients G, H, g, h are to be determined. They must satisfy the following equations:

$$G + g = B, \tag{3.3}$$

$$A(H+h) + Gg = AC, \tag{3.4}$$

$$Gh + gH = AD, \tag{3.5}$$

$$Hh = AE. \tag{3.6}$$

Denoting

$$Gg = Ax \tag{3.7}$$

and supposing that x is known, we can find G and g from (3.3) and (3.7):

$$G = \frac{B + \sqrt{(B^2 - 4Ax)}}{2}, \quad g = \frac{B - \sqrt{(B^2 - 4Ax)}}{2}, \tag{3.8a}$$

and then (3.4) and (3.5) yield

$$H = \frac{C-x}{2} + \frac{B(C-x) - 2AD}{2\sqrt{(B^2 - 4Ax)}}, \quad h = \frac{C-x}{2} - \frac{B(C-x) - 2AD}{2\sqrt{(B^2 - 4Ax)}}. \tag{3.8b}$$

Equation (3.6) has not yet been used, but it must also be satisfied. Substituting (3.8b) into (3.6), and simplifying, we obtain

$$x^3 - 2Cx^2 + (C^2 + BD - 4AE)x - (BCD - B^2E - AD^2) = 0. \tag{3.9}$$

This is the *resolvent cubic*. It may be solved as shown in section 2, yielding three roots x_1, x_2, x_3 and, substituting those in (3.8), we may find three alternative sets of coefficients G, H, g, h, so that the quartic can be factorized in three different ways. All three sets of coefficients can only be real if all roots of the quartic are real, and then all roots of the resolvent cubic must also be real. The inverse is not always true: the resolvent cubic will have three real roots also when all roots of the quartic are complex. To understand all cases which may occur, let us assume that one set of real coefficients, say G_1, H_1, g_1, h_1, is known (*one* such set must, of course, always exist). We have then $x_1 = G_1 g_1/A$, and the remaining roots x_2, x_3 of (3.9) are easily shown to be

$$x_{2,3} = H_1 + h_1 + \frac{G_1 g_1}{2A} \pm \frac{\sqrt{[(G_1^2 - 4AH_1)(g_1^2 - 4Ah_1)]}}{2A}, \tag{3.10}$$

whence

$$B^2 - 4Ax_{2,3} = [\sqrt{(G_1^2 - 4AH_1)} \mp \sqrt{(g_1^2 - 4Ah_1)}]^2. \tag{3.11}$$

It is seen now that three basic cases (excluding multiple roots) may occur:

(a) The quartic has four real roots; $(G_1^2 - 4AH_1)$ and $(g_1^2 - 4Ah_1)$ are both positive. The cubic has three real roots, each smaller than $B^2/4A$, and three alternative factorizations with real coefficients exist.

(b) The quartic has two real and two complex roots; one of the quantities $(G_1^2 - 4AH_1)$ and $(g_1^2 - 4Ah_1)$ is negative. The cubic has only one real root, necessarily smaller than $B^2/4A$, and only one factorization with real coefficients exists.

(c) The quartic has four complex roots; $(G_1^2-4AH_1)$ and $(g_1^2-4Ah_1)$ are both negative. The cubic has three real roots, of which only the smallest one is less than $B^2/4A$, and only one factorization with real coefficients exists.

In practice, one factorization only is needed, and the smallest (or the only existing one) real root of the resolvent cubic will naturally be used.

The above method may fail only if one root of the resolvent cubic is *equal*† to $B^2/4A$. In this case, however, the quartic can be factorized in a particularly simple way, avoiding the cubic resolvent, as shown in section 3.2—some other interesting particular cases, including those of multiple roots, are also considered there.

3.2 SPECIAL CASES

(i) *One root of the resolvent cubic equals $B^2/4A$*

Substituting $B^2/4A$ for x in (3.9), we find

$$C = \frac{B^2}{4A}+\frac{2AD}{B}. \tag{3.13}$$

---

† The method might seem awkward when only one factorization with real coefficients exists in which G and g are very nearly equal. It is then seen from (3.8a) that (B^2-4Ax) must be very small, so that it must be calculated with many accurate decimals to ensure a sufficient accuracy of G, g, and particularly H, h. However, in such a case, it will be better to determine H, h first from (3.4) and (3.6):

$$H, h = \frac{\mathrm{C}-x\pm\sqrt{[(\mathrm{C}-x)^2-4\mathrm{A}E]}}{2}, \tag{3.12a}$$

then, from (3.3) and (3.5)

$$G, g = \frac{B}{2}\pm\frac{B(\mathrm{C}-x)-2AD}{2\sqrt{[(\mathrm{C}-x)^2-4AE]}}, \tag{3.12b}$$

and these formulae should then be used, rather than (3.8a, b).

As to the case when both $g \approx G$ and $h \approx H$, see section 3.2 (ix).

In this case, the use of resolvent cubic may be awkward: the root $B^2/4A$, on substitution in (3.8b), leads to indeterminate forms, and the two remaining roots may sometimes be complex or, even if real, may give complex values of coefficients (3.8). However, the factorization in real terms is, of course, always possible and, in this particular case, can be performed very simply, avoiding the cubic resolvent, as follows.

If $E < AD^2/B^2$, we have

$$f(\lambda) = A\left\{\lambda^2+\frac{B}{2A}\lambda+\left[\frac{D}{B}+\sqrt{\left(\frac{D^2}{B^2}-\frac{E}{A}\right)}\right]\right\}\left\{\lambda^2+\frac{B}{2A}\lambda+\left[\frac{D}{B}-\sqrt{\left(\frac{D^2}{B^2}-\frac{E}{A}\right)}\right]\right\}, \tag{3.14}$$

as can be checked directly. The four roots may be all real, or all complex, or two real and two complex. If all are complex, they have identical real parts.

If $E > AD^2/B^2$, there are complex terms in (3.14) and, in this case, the most convenient factorization is

$$f(\lambda) = A[(\lambda+R_1)^2+J^2][(\lambda+R_2)^2+J^2], \tag{3.15}$$

where

$$\left.\begin{aligned} 4AR_{1,2} &= B\pm\sqrt{\left\{\frac{B^2}{2}-\frac{8A^2D}{B}\right.} \\ &\qquad \left.+\sqrt{\left[\left(\frac{B^2}{2}-\frac{8A^2D}{B}\right)^2+64A^3\left(E-\frac{AD^2}{B^2}\right)\right]}\right\} \\ 4AJ &= \sqrt{\left\{\frac{8A^2D}{B}-\frac{B^2}{2}\right.} \\ &\qquad \left.+\sqrt{\left[\left(\frac{B^2}{2}-\frac{8A^2D}{B}\right)^2+64A^3\left(E-\frac{AD^2}{B^2}\right)\right]}\right\}. \end{aligned}\right\} \tag{3.16}$$

This can again be checked directly. All four roots are complex: $(-R_1\pm\mathrm{i}J)$ and $(-R_2\pm\mathrm{i}J)$, the imaginary parts being identical.

Finally, if $E = AD^2/B^2$, the quartic factorizes as a perfect square:

$$f(\lambda) = A\left(\lambda^2+\frac{B}{2A}\lambda+\frac{D}{B}\right)^2. \tag{3.17}$$

It may be mentioned that, if $E = AD^2/B^2$ while (3.13) is not true, then a factorization with $H = h = AD/B$ exists, and the resolvent cubic is not needed, see also paragraph (viii) below.

(ii) *Double roots*

It is obvious that, if the quartic has *one double root* (which must be real), then the resolvent cubic must also have one. In this case, there exists a factorization of the form (3.2), where one of the factors is a perfect square. If the set of coefficients is then G_1, H_1, g_1, h_1, then one of the expressions $(G_1^2-4AH_1)$ and $(g_1^2-4Ah_1)$ must be 0, so that, from (3.10) we find $x_2 = x_3$. It is then the third (unequal) root which, on substitution into (3.8), leads to the factorization exhibiting the factor with double root. The condition, which must be satisfied by the coefficients of the quartic, is that $\Delta = \pm 1$ for the resolvent cubic (3.9) and, using (2.11), this is easily presented in the form

$$\begin{aligned}(2C^3+27AD^2+27B^2E-9BCD-72ACE)^2&\\ = 4(C^2-3BD+12AE)^3&\end{aligned} \tag{3.18}$$

or, expanding and simplifying,

$$\left.\begin{aligned}&256A^3E^3-(27B^4+128A^2C^2+192A^2BD-144AB^2C)E^2\\ &\quad+2(8AC^4+72A^2CD^2+9B^3CD-40ABC^2D\\ &\quad-3AB^2D^2-2B^2C^3)E-D^2(4AC^3+4B^3D\\ &\quad+27A^2D^2-18ABCD-B^2C^2)=0.\end{aligned}\right\} \tag{3.19}$$

This condition is somewhat complicated, but it may be useful in some particular cases.

The case of *two double roots* was encountered already under (i), and the conditions are:

$$C = \frac{B^2}{4A}+\frac{2AD}{B} \quad \text{and} \quad E = \frac{AD^2}{B^2}. \tag{3.20}$$

It is seen easily that (3.18) is then identically satisfied. The factorization is given by (3.17), and the two double roots are either both real or both complex, depending on whether

$$B^2 \gtrless \frac{16A^2D}{B}. \tag{3.21}$$

A particular simple case occurs when

$$D = \frac{B^3}{16A^2}, \quad \text{thus} \quad C = \frac{3B^2}{8A}, \; E = \frac{B^4}{256A^3}; \tag{3.22}$$

the quartic then possesses a *quadruple root* and factorizes as follows:

$$f(\lambda) = A\left(\lambda+\frac{B}{4A}\right)^4. \tag{3.23}$$

(iii) *Triple root*

If the quartic has a triple root (which must be real), then the resolvent cubic must obviously also have one, so that (3.9) becomes a perfect cube. The conditions, which must be satisfied by the coefficients of (3.1), are obtained by assuming that $b^2-3ac = 0$, and $b^3-27a^2d = 0$ for the cubic (3.9), whence

$$C^2-3BD+12AE = 0, \quad 8C^3-27BCD+27B^2E+27AD^2 = 0. \tag{3.24}$$

The relationship (3.17) is then identically satisfied. The conditions (3.22) can be brought to the form

$$D = \frac{B(4AC-B^2)\pm(B^2-\frac{8}{3}AC)^{3/2}}{8A^2}, \quad E = \frac{BD}{4A}-\frac{C^2}{12A}. \tag{3.25}$$

The factorization of the quartic is very simple in this case, the only (triple) root of the resolving cubic being $x = 2C/3$. It is even possible to derive very simple formulae for the roots of our quartic, viz.

$$\lambda_{1,2,3} = \frac{-B \pm \sqrt{(B^2 - \frac{8}{3}AC)}}{4A}, \quad \lambda_4 = \frac{-B \mp 3\sqrt{(B^2 - \frac{8}{3}AC)}}{4A}, \tag{3.26}$$

where upper or lower signs should be taken depending on whether

$$D \gtrless \frac{B(4AC - B^2)}{8A^2}. \tag{3.27}$$

It is obvious that $(B^2 - \frac{8}{3}AC) \geqslant 0$, otherwise D could not be real, cf. (3.25). If $B^2 = 8AC/3$, we obtain again the relations (3.22), the case of a quadruple root.

(iv) *Resolvent cubic has one zero root*

This case occurs when the constant term of (3.9) is 0, i.e. when

$$\mathscr{R} = BCD - B^2E - AD^2 = 0. \tag{3.28}$$

Equation (3.1) then factorizes immediately in the form

$$f(\lambda) = \left[A\lambda^2 + B\lambda + \left(C - \frac{AD}{B}\right)\right]\left(\lambda^2 + \frac{D}{B}\right). \tag{3.29}$$

The expression of (3.28) is the known Hurwitz–Routh discriminant for oscillatory stability. There are two equal and opposite roots.

(v) *Resolvent cubic has a double zero root*

This happens when the constant term and the coefficient of x in (3.9) are both 0, and these conditions may be written in the form

$$C = \frac{2AD}{B} \pm \sqrt{(-BD)}, \quad E = \frac{C^2 + BD}{4A}. \tag{3.30}$$

B and D must have opposite signs. All four roots are real, and one of the equal and opposite roots is double. The factorization (3.29) still applies, but in this case there exist very simple formulae for all roots of the quartic, viz.

if $C > \dfrac{2AD}{B}$ then

$$\lambda_{1,2} = -\frac{\sqrt{(-BD)}}{B},$$

$$\lambda_3 = \frac{\sqrt{(-BD)}}{B}, \quad \lambda_4 = -\frac{B}{A} + \frac{\sqrt{(-BD)}}{B}, \tag{3.31a}$$

if $C < \dfrac{2AD}{B}$ then

$$\lambda_{1,2} = \frac{\sqrt{(-BD)}}{B},$$

$$\lambda_3 = -\frac{\sqrt{(-BD)}}{B}, \quad \lambda_4 = -\frac{B}{A} - \frac{\sqrt{(-BD)}}{B}. \tag{3.31b}$$

(vi) *Resolvent cubic has a triple zero root*

This simple case occurs only when

$$C = 0, \quad D = -\frac{B^3}{4A^2}, \quad E = -\frac{B^4}{16A^3}, \tag{3.32}$$

and then the quartic has a triple root and an equal and opposite single root

$$\lambda_{1,2,3} = -\frac{B}{2A}, \quad \lambda_4 = \frac{B}{2A}. \tag{3.33}$$

(viii) *Resolvent cubic has one zero root and a double root*

This is another simple case which occurs only when

$$C = \frac{AD}{B} + \frac{B^2}{4A} \quad \text{and} \quad E = \frac{BD}{4A}. \tag{3.34}$$

The quartic then possesses one double root and two equal and opposite roots:

$$\lambda_{1,2} = -\frac{B}{2A}, \quad \lambda_{3,4} = \pm\sqrt{\left(-\frac{D}{B}\right)}. \tag{3.35}$$

(viii) *Case of quasi-symmetrical quartic*

Suppose $D = Bk$ and $E = Ak^2$, so that (3.1) has the form

$$f(\lambda) = A\lambda^4 + B\lambda^3 + C\lambda^2 + Bk\lambda + Ak^2 = 0 \tag{3.36}$$

(this becomes a symmetrical quartic equation if $k = 1$, and an anti-symmetrical one if $k = -1$).

We assume again the factorization (3.2), but equations (3.3) to (3.6) now become

$$G + g = B, \tag{3.37}$$

$$A(H+h) + Gg = AC, \tag{3.38}$$

$$Gh + gH = ABk, \tag{3.39}$$

$$Hh = A^2k^2. \tag{3.40}$$

From (3.37), (3.39) and (3.40) we have

$$\left(\frac{Gh+gH}{G+g}\right)^2 = Hh, \quad \text{or} \quad (H-h)(G^2h - g^2H) = 0. \tag{3.41}$$

Assume first $H = h$, then, from (3.37), (3.38) and (3.39),

$$H = h = Ak, \quad Gg = A(C-2Ak), \tag{3.42}$$

and hence

$$G = \frac{B + \sqrt{[B^2 - 4A(C-2Ak)]}}{2}, \quad g = \frac{B - \sqrt{[B^2 - 4A(C-2Ak)]}}{2}. \tag{3.43}$$

Thus, *a factorization* has been found without using the resolvent cubic, and this could be obtained even more simply by introducing an auxiliary variable $(\lambda + k/\lambda)$ in (3.36).

The expressions of (3.43) give real or complex values for G, g depending on whether

$$B^2 \gtrless 4A(C-2Ak). \tag{3.44}$$

In the second case, to obtain a factorization with real coefficients, we must make the *second assumption* $G^2h = g^2H$, whence, by using (3.39) and (3.37),

$$H = Ak\frac{G}{g}, \quad h = Ak\frac{g}{G}, \tag{3.45}$$

whereupon (3.38) becomes

$$(Gg)^2 - A(2Ak+C)(Gg) + A^2B^2k = 0. \tag{3.46}$$

Hence

$$Gg = \frac{A(2Ak+C) \mp A\sqrt{[(2Ak+C)^2 - 4B^2k]}}{2}, \tag{3.47}$$

and then, from (3.37) and (3.47),

$$\left.\begin{aligned} G &= \frac{B + \sqrt{\{B^2 - 2A(2Ak+C) \pm 2A\sqrt{[(2Ak+C)^2 - 4B^2k]}\}}}{2}, \\ g &= \frac{B - \sqrt{\{B^2 - 2A(2Ak+C) \pm 2A\sqrt{[(2Ak+C)^2 - 4B^2k]}\}}}{2}, \end{aligned}\right\} \tag{3.48}$$

so that *two alternative factorizations* have been obtained, as given by (3.48) and (3.45), and again without using the resolving cubic.

If the upper sign applies in the inequality (3.44), then the first factorization has real coefficients, and the two alternatives have either both real or both complex coefficients, depending on whether

$$(2Ak+C)^2 \gtrless 4B^2k, \tag{3.49}$$

but, in either case, the first factorization is more convenient.

If, however, the lower sign applies in (3.44), then the first factorization has complex coefficients, hence one (and obviously only one) of the alternative factorizations must have real ones, and it will be the one with upper signs in (3.47) and (3.48).

If finally,

$$B^2 = 4A(C-2Ak) \quad\text{or}\quad C = 2Ak+\frac{B^2}{4A}, \tag{3.50}$$

then the quartic factorizes as a perfect square:

$$f(\lambda) = A\left(\lambda^2+\frac{B}{2A}\lambda+k\right)^2. \tag{3.51}$$

In all cases, the resolvent cubic (3.9) assumes the following form, which factorizes immediately:

$$\begin{aligned} x^3-2Cx^2+(C^2+B^2k-4A^2k^2)x-B^2k(C-2Ak) \\ = [x-(C-2Ak)][x^2-(C+2Ak)x+B^2k] = 0. \end{aligned} \tag{3.52}$$

The first factor leads to (3.42) and thus to the first factorization of the quartic. The second factor leads to (3.47) and to the two alternative solutions.

(ix) *Critical case of a quartic being nearly a perfect square*

It was seen above [case (i)] that the quartic factorizes as a perfect square when $C = B^2/4A+2AD/B$ and $E = AD^2/B^2$, and then $G = g = \frac{1}{2}B$, $H = h = AD/B$. The case when $g \approx G$ and $h \approx H$ may prove awkward if there are complex roots so that only one factorization with real coefficients exists. In such a case the only positive values of (B^2-4Ax) in (3.8), and of $[(C-x)^2-4AE]$ in (3.11) are very small, and either must be calculated with a great number of accurate decimals to ensure a moderate accuracy of G, g, H, h. The resolvent cubic of such an "ill-conditioned" quartic has its Δ very nearly equal to ± 1, and the method of Appendix 2 must be used to solve this cubic. The procedure works but may be very inconvenient, and an alternative method of direct factorization without a resolvent cubic is suggested here. Let

$$C = \frac{B^2}{4A}+\frac{2AD}{B}+c_1, \quad E = \frac{AD^2}{B^2}+e_1, \tag{3.53}$$

where c_1, e_1 are supposed small, and let us assume

$$\left.\begin{aligned} G = \tfrac{1}{2}B+\varepsilon, \qquad g = \tfrac{1}{2}B-\varepsilon, \\ H = \frac{AD}{B}+G\eta, \quad h = \frac{AD}{B}-g\eta, \end{aligned}\right\} \tag{3.54}$$

where ε and η are unknown small numbers, to be determined. It is seen that the conditions (3.3) and (3.5) are identically satisfied, and (3.4) and (3.6) lead to

$$2\varepsilon\eta-\frac{\varepsilon^2}{A} = c_1, \tag{3.55a}$$

$$2\varepsilon\eta\frac{AD}{B}-\eta^2\left(\frac{B^2}{4}-\varepsilon^2\right) = Ae_1, \tag{3.55b}$$

whence, eliminating terms not containing ε or η,

$$e_1\varepsilon^2-2A\left(e_1-\frac{D}{B}c_1\right)\varepsilon\eta-Kc_1\eta^2 = 0, \tag{3.56}$$

where

$$K = \tfrac{1}{4}B^2-\varepsilon^2. \tag{3.57}$$

The ratio η/ε may then be determined from (3.56), taking first $K = \frac{1}{4}B^2$,

$$K\frac{\eta}{\varepsilon} = A\left(\frac{D}{B}-\frac{e_1}{c_1}\right)\pm\sqrt{\left[A^2\left(\frac{D}{B}-\frac{e_1}{c_1}\right)^2+K\frac{e_1}{c_1}\right]}, \tag{3.58}$$

whereupon, from (3.55a),

$$\varepsilon = \sqrt{\left(\frac{Ac_1}{2A\eta/\varepsilon-1}\right)}. \tag{3.59}$$

The accuracy may be improved, if needed, by correcting K according to (3.57).

This method works very well provided c_1 and e_1 are computed with a sufficient number of decimals so as to determine the ratio e_1/c_1 with any required accuracy. If it happens, however, that c_1 is much smaller than e_1, then it will be more convenient to solve (3.56) for the ratio ε/η:

$$\frac{\varepsilon}{\eta} = A\left(\frac{D}{B}-\frac{c_1}{e_1}\right) \pm \sqrt{\left[A^2\left(\frac{D}{B}-\frac{c_1}{e_1}\right)^2 + K\frac{c_1}{e_1}\right]}, \tag{3.60}$$

and continue as before.

EXAMPLES

(4.1) $4x^3-20x^2+17x+14 = 0$

$$b^2-3ac = 196, \quad \sqrt{(b^2-3ac)} = 14, \quad \Delta = 0{\cdot}41691$$

$$y_1 = 0{\cdot}92857, \quad y_2 = -0{\cdot}14286, \quad y_3 = -0{\cdot}78571$$

$$x_1 = -0{\cdot}50000, \; x_2 = 2{\cdot}00001, \quad x_3 = 3{\cdot}49999.$$

The exact roots are: $-0{\cdot}5$, 2, $3{\cdot}5$.

(4.2) $4x^3+30x^2+78x-47 = 0$

$$b^2-3ac = -36, \quad \sqrt{(3ac-b^2)} = 6, \quad \Delta = -117$$

$$z_1 = -3, \quad z_{2,3} = 1{\cdot}5 \mp 2{\cdot}73861i$$

$$x_1 = 0{\cdot}5, \quad x_{2,3} = -4 \pm 2{\cdot}73861i.$$

(4.3) $12x^3+20x^2+11x+2 = 0$

$$b^2-3ac = 4, \quad \Delta = 1.$$

The equation has a double root and, from (2.12),

$$x_1 = -\tfrac{2}{3}, \quad x_{2,3} = -\tfrac{1}{2}.$$

(4.4) $\lambda^4+6\lambda^3-8\lambda^2-22\lambda-105 = 0$

The resolvent cubic is $x^3+16x^2+352x-4352 = 0$,

$$b^2-3ac = -800, \quad \sqrt{(3ac-b^2)} = 28{\cdot}284271, \quad \Delta = -3{\cdot}535534.$$

From Table 2 the only real root is $z_1 = -0{\cdot}70710$, therefore $x_1 = 7{\cdot}99987$ or, as easily checked, exactly 8. Equations (3.8) then give

$$G, g = 3 \pm 1, \quad H, h = -8 \mp 13,$$

and the quartic factorizes

$$(\lambda^2+4\lambda-21)(\lambda^2+2\lambda+5) = 0,$$

the roots being

$$\lambda_1 = 3, \quad \lambda_2 = -7, \quad \lambda_{3,4} = -1 \pm 2\mathrm{i}.$$

(4.5) $9\lambda^4+60\lambda^3+202\lambda^2+340\lambda+301 = 0$

We have here

$$C = \frac{B^2}{4A} + \frac{2AD}{B},$$

therefore the roots are obtained from (3.16)

$$36R_{1,2} = 60 \pm 12; \quad R_1 = 2, R_2 = \tfrac{4}{3}; \quad 36J = \surd 3888, J = \surd 3;$$

$$\lambda_{1,2} = -2 \pm \mathrm{i}\surd 3, \quad \lambda_{3,4} = -\tfrac{4}{3} \pm \mathrm{i}\surd 3.$$

The quartic factorized is $(\lambda^2+4\lambda+7)(9\lambda^2+24\lambda+43) = 0$.

(4.6) $\lambda^4+4{\cdot}02\lambda^3+29{\cdot}04\lambda^2+50{\cdot}1\lambda+100 = 0$

The resolvent cubic is

$$x^3-58{\cdot}08x^2+644{\cdot}736x-1722{\cdot}66408 = 0,$$

$$b^2-3ac = 1439{\cdot}1156, \quad \surd(b^2-3ac) = 37{\cdot}935677,$$

$$\Delta = -0{\cdot}928154.$$

Table 1 gives three real roots for y, and the greatest of them, leading to the smallest x and to the only factorization in real terms, is

$$y = 0{\cdot}60576, \quad \text{hence} \quad x = 4{\cdot}04005.$$

This will give $B^2-4Ax = 0{\cdot}00020$ which is a very small difference of two large numbers, and therefore inaccurate. We have here an obvious case for using (3.12) in preference to (3.8), and we obtain

$$(C-x)^2-4AE = 224{\cdot}99750, \quad \surd[(C-x)^2-4AE] = 14{\cdot}999917,$$

$$H = 19{\cdot}99993, \quad h = 5{\cdot}00002, \quad G = 2{\cdot}01999, \quad g = 2{\cdot}00001.$$

The exact factorization in this case is

$$(\lambda^2+2{\cdot}02\lambda+20)(\lambda^2+2\lambda+5) = 0,$$

and it is seen that the exact value of x is 4·04. The value of x found above has an erroneous last decimal, this being clearly due to the table giving only 5 decimals for y.

(4.7) $\lambda^4+4{\cdot}02\lambda^3+14{\cdot}09\lambda^2+20{\cdot}2\lambda+25{\cdot}25 = 0$

This example has been taken from Reference 9 and illustrates the case when the quartic is very nearly a perfect square. The resolvent cubic is

$$x^3-28{\cdot}18x^2+178{\cdot}7321x-328{\cdot}07426 = 0.$$

$$b^2-3ac = 257{\cdot}9161, \quad \surd(b^2-3ac) = 16{\cdot}059766499,$$

$$\Delta = -\frac{32{\cdot}119532988}{32{\cdot}119532998} = -1+0{\cdot}000'000'000'31,$$

hence, using Appendix 2,

$$y_1 = -1+0{\cdot}000'000'000'03, \quad y_2 = 0{\cdot}5000072, \quad y_3 = 0{\cdot}4999928,$$

$$x_2 = 4{\cdot}0400007$$

and, from (3.8),

$$G = 2{\cdot}01996, \quad g = 2{\cdot}00004, \quad H = 5{\cdot}05002, \quad h = 4{\cdot}99998.$$

The exact factorization is $(\lambda^2+2{\cdot}02\lambda+5{\cdot}05)(\lambda^2+2\lambda+5)$, so that our results are accurate enough, but it was necessary to calculate Δ with eleven decimals, and the computation was rather awkward.

Using the method of section 3.2 (ix) we obtain

$$e_1 = 0{\cdot}0006249845, \quad c_1 = 0{\cdot}0001487562,$$

$$e_1/c_1 = 4{\cdot}20140, \quad D/B = 5{\cdot}02488.$$

From (3.58, 59) $\eta/\varepsilon = 1{\cdot}24376$, $\varepsilon = 0{\cdot}01000$, $G = 2{\cdot}02000$, $g = 2{\cdot}00000$, $H = 5{\cdot}05000$, $h = 5{\cdot}00000$. It was necessary to calculate c_1, e_1 with 10 decimals to obtain results of such accuracy, but this was very much easier than solving a cubic with similar accuracy, and the computation presented no difficulties whatsoever.

(4.8) *Probstein's*$^{(15)}$ *quartic equation for inversion of Prandtl–Meyer relation in the case* $\gamma = 5/4$ *is*

$$\zeta^4 - 2\zeta^2 - \tfrac{8}{3}\mu\zeta - \tfrac{1}{3} = 0 \quad \left(\mu = \cot\frac{\nu}{3} \geqslant \frac{1}{\sqrt{3}}\right).$$

The resolvant cubic is $x^3 + 4x^2 + \frac{16}{3}x + \frac{64}{9}\mu^2 = 0$, therefore $b^2 - 3ac = 0$, and (2.8) gives

$$x = -4r^2, \quad \text{where} \quad r^2 = \frac{\sqrt[3]{(3\mu^2-1)}+1}{3}.$$

Formulae (3.8) now yield

$$G, g = \pm 2r, \quad H, h = 2r^2 - 1 \pm \frac{2\mu}{3r},$$

so that the given quartic factorizes as follows:

$$\left[\zeta^2 + 2r\zeta + \left(2r^2 - 1 + \frac{2\mu}{3r}\right)\right]\left[\zeta^2 - 2r\zeta + \left(2r^2 - 1 - \frac{2\mu}{3r}\right)\right] = 0.$$

The first factor has complex roots, and the only real positive root is obtained from the second factor:

$$\zeta = r + \sqrt{\left(1 - r^2 + \frac{2\mu}{3r}\right)}.$$

This solution is simpler than that obtained by Probstein who used Euler's method for solving the quartic.

(4.9) *Prof. Collar's*$^{(16)}$ *cubic equation for plane oblique shock waves*

We consider three examples of paragraph 4 of Reference 16:

(a) $x^3 + 1{\cdot}3x^2 - 9x + 0{\cdot}3 = 0$

$$b^2 - 3ac = 28{\cdot}69, \quad \sqrt{(b^2 - 3ac)} = 5{\cdot}356305, \quad \Delta = 0{\cdot}38326.$$

From Table 1 the larger negative root is $y_3 = -0{\cdot}79322$, hence $x_3 = 2{\cdot}39915$.

(b) $x^3 + 2{\cdot}07401x^2 - 4{\cdot}6169x + 0{\cdot}56896 = 0$

$$b^2 - 3ac = 18{\cdot}15222, \quad \sqrt{(b^2 - 3ac)} = 4{\cdot}26054, \quad \Delta = 0{\cdot}77183.$$

The larger negative root is $y_3 = -0{\cdot}68404$, hence $x_3 = 1{\cdot}25159$.

(c) $x^3 + 13x^2 - 9x + 3 = 0$

$$b^2 - 3ac = 196, \quad \sqrt{(b^2 - 3ac)} = 14, \quad \Delta = 1{\cdot}00729.$$

The only real root y_1 is positive. There is no positive root for x.

(4.10) *F. A. Lee's*$^{(17)}$ *quartic equation of special flutter analysis*

The numerical example is $\lambda^4 + \lambda^2 + \lambda - 1 = 0$.

The resolvent cubic becomes $x^3 - 2x^2 + 5x - 1 = 0$,

$$b^2 - 3ae = -11, \quad \sqrt{(3ac - b^2)} = 3{\cdot}316625, \quad \Delta = 1{\cdot}38421.$$

From Table 2 the only real root is $z_1 = 0{\cdot}38520$, therefore $x_1 = -0{\cdot}18504$, and from (3.8)

$$G = -g = 0{\cdot}43016, \quad H = -0{\cdot}56984, \quad h = 1{\cdot}75488.$$

The quartic factorizes:

$$(\lambda^2 + 0{\cdot}43016\lambda - 0{\cdot}56984)(\lambda^2 - 0{\cdot}43016 + 1{\cdot}75488) = 0,$$

and the roots are

$$\lambda_1 = 0{\cdot}56984, \quad \lambda_2 = -1, \quad \lambda_{3,4} = 0{\cdot}21508 \pm 1{\cdot}30714i.$$

The procedure is more rapid, and the results more accurate than those given by Lee. His resolvent cubic is a special case of ours, but he solves it by Cardano's formulae.

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APPENDIX I

DETAILS OF COMPUTING TABLES 1 AND 2

(A) SOLUTION OF CUBIC (2.4)

(A1) *Case $\Delta > 1$. Solution by hyperbolic functions*

The only real root $y_1 > 1$ (see Fig. 1), so we may write

$$y_1 = \cosh\alpha, \tag{I.1}$$

whereupon (2.4) becomes

$$4\cosh^3\alpha - 3\cosh\alpha = \cosh 3\alpha = \Delta, \tag{I.2}$$

and hence

$$y_1 = \cosh\frac{\cosh^{-1}\Delta}{3}. \tag{I.3}$$

The cubic now factorizes

$$4y^3 - 3y - \Delta = (y - y_1)[4y^2 + 4y_1 y + (4y_1^2 - 3)] = 0, \tag{I.4}$$

and the two complex roots become

$$y_{2,3} = -0{\cdot}5y_1 \pm i\surd[0{\cdot}75(y_1^2 - 1)] \tag{I.5}$$

or

$$y_{2,3} = \tfrac{1}{2}(-\cosh\alpha \pm i\surd 3\,.\,\sinh\alpha). \tag{I.5a}$$

Alternative ("Cardano") solution by square and cube roots

Assume

$$y_1 = \tfrac{1}{2}(u + v), \tag{I.6}$$

where u and v are supposed real and

$$uv = 1, \tag{I.7}$$

then equation (2.4) becomes

$$u^3+v^3 = 2\Delta, \tag{I.8}$$

and hence

$$u = \sqrt[3]{[\Delta+\sqrt{(\Delta^2-1)}]}, \quad v = \sqrt[3]{[\Delta-\sqrt{(\Delta^2-1)}]}, \tag{I.9}$$

so that

$$y_1 = \tfrac{1}{2}\{\sqrt[3]{[\Delta+\sqrt{(\Delta^2-1)}]}+\sqrt[3]{[\Delta-\sqrt{(\Delta^2-1)}]}\}. \tag{I.10}$$

The complex roots become, from (I.5),

$$y_{2,3} = \tfrac{1}{4}[-(u+v)\pm i(u-v)\sqrt{3}]. \tag{I.11}$$

(A2) *Case $\Delta < 1$. Solution by trigonometric functions*

There are three real roots and, if y_1 denotes the only positive one, then (see Fig. 1) $\frac{1}{2}\sqrt{3} < y_1 < 1$, so we may write

$$y_1 = \cos\beta, \tag{I.12}$$

where $0 < \beta < 30°$. Equation (2.4) now becomes

$$4\cos^3\beta-3\cos\beta = \cos 3\beta = \Delta, \tag{I.13}$$

and hence

$$y_1 = \cos\frac{\cos^{-1}\Delta}{3}. \tag{I.14}$$

The cubic factorizes as before (I.4), and the two negative roots become

$$y_{2,3} = -\tfrac{1}{2}\cos\beta\pm\tfrac{1}{2}\sqrt{3}\,.\,\sin\beta = -\cos(60°\pm\beta). \tag{I.15}$$

The "Cardano" solution (I.10) and (I.11) involves cube roots of complex quantities in this case, and is therefore inconvenient (*casus irreducibilis*).

(B) SOLUTION OF CUBIC (2.7)

Solution by hyperbolic functions

The only real root z_1 (see Fig. 2) may have any value, so we can write

$$z_1 = \sinh\gamma, \tag{I.16}$$

whereupon (2.7) becomes

$$4\sinh^3\gamma+3\sinh\gamma = \sinh 3\gamma = \Delta, \tag{I.17}$$

and hence

$$z_1 = \sinh\frac{\sinh^{-1}\Delta}{3}. \tag{I.18}$$

The cubic factorizes

$$4z^3+3z-\Delta = (z-z_1)[4z^2+4z_1z+(4z_1^2+3)] = 0, \tag{I.19}$$

and the two complex roots become

$$z_{2,3} = -0{\cdot}5z_1\pm i\sqrt{[0{\cdot}75(z_1^2+1)]} \tag{I.20}$$

or

$$z_{2,3} = \tfrac{1}{2}(-\sinh\gamma\pm i\sqrt{3}\,.\,\cosh\gamma). \tag{I.20a}$$

Alternative ("Cardano") solution by square and cube roots

Assume

$$z_1 = \tfrac{1}{2}(s-t), \tag{I.21}$$

where s and t are real, and

$$st = 1, \tag{I.22}$$

then (2.7) becomes

$$s^3-t^3 = 2\Delta \tag{I.23}$$

and hence

$$s = \sqrt[3]{[\sqrt{(\Delta^2+1)}+\Delta]}, \quad t = \sqrt[3]{[\sqrt{(\Delta^2+1)}-\Delta]}, \tag{I.24}$$

so that

$$z_1 = \tfrac{1}{2}\{\sqrt[3]{[\sqrt{(\Delta^2+1)}+\Delta]} - \sqrt[3]{[\sqrt{(\Delta^2+1)}-\Delta]}\}. \qquad (I.25)$$

The complex roots become, from (I.20),

$$z_{2,3} = \tfrac{1}{4}[-(s-t)\pm i(s+t)\sqrt{3}]. \qquad (I.26)$$

The above formulae are sufficient for computing tables 1 and 2 to any number of decimals provided the tables of hyperbolic and trigonometric functions (and alternatively those of square and cube roots) with sufficient number of decimals are available. However, it was found in practice more convenient, once a few entries for y_1 (or z_1) had been determined and first differences written down, to find subsequent values by extrapolation and check the last decimal by trial and error (substituting in the original cubic equation), until the error became less than ½ of 0·00001. The remaining two roots were then determined from (I.5) or (I.20) but, if Δ was near to 1 in (2.4), y_1 had to be determined to six or seven decimals to ensure five correct decimals in y_2, y_3. The computation was very rapid. Several checks have been applied.

APPENDIX II

POWER SERIES FOR ROOTS OF CUBIC (2.4) WITH Δ NEAR TO 1

LET Δ be slightly greater than 1, so that we may write

$$\Delta = 1+9\eta, \qquad (II.1)$$

where η is small, then y_1 will also be slightly greater than 1, say

$$y_1 = 1+\alpha. \qquad (II.2)$$

Substituting (II.1) and (II.2) into (2.4), we obtain

$$\alpha = \eta - \frac{4}{3}\alpha^2 - \frac{4}{9}\alpha^3, \qquad (II.3)$$

and then, by successive substitutions, the following series is obtained:

$$y_1 = 1+\eta-\frac{4}{3}\eta^2+\frac{28}{9}\eta^3-\frac{80}{9}\eta^4+\frac{2288}{81}\eta^5\ldots \qquad (II.4)$$

and, on substituting this in (I.5), and expanding,

$$y_{2,3} = -0{\cdot}5y_1$$

$$\pm i\left(1-\frac{5}{12}\eta+\frac{77}{96}\eta^2-\frac{2431}{1152}\eta^3+\frac{1062347}{165888}\eta^4\ldots\right)\sqrt{(1{\cdot}5\eta)}. \qquad (II.5)$$

Similarly, if Δ is slightly smaller than 1

$$\Delta = 1-9\varepsilon,$$

we obtain, by replacing η by $(-\varepsilon)$ in (II.4) and (II.5),

$$y_1 = 1-\varepsilon-\frac{4}{3}\varepsilon^2-\frac{28}{9}\varepsilon^3-\frac{80}{9}\varepsilon^4-\frac{2288}{81}\varepsilon^5\ldots \qquad \text{(II.6)}$$

and

$$y_{2,3} = -0{\cdot}5y_1$$

$$\pm\left(1+\frac{5}{12}\varepsilon+\frac{77}{96}\varepsilon^2+\frac{2431}{1152}\varepsilon^3+\frac{1062347}{165888}\varepsilon^4\ldots\right)\sqrt{(1{\cdot}5\varepsilon)}. \quad \text{(II.7)}$$

If, for example, $\Delta = 0{\cdot}91$, we have $\varepsilon = 0{\cdot}01$, and

$$y_1 = 1-0{\cdot}01-0{\cdot}0001333\ldots-0{\cdot}000003111-0{\cdot}000000088\ldots$$
$$= 0{\cdot}98986,$$

$$y_{2,3} = -0{\cdot}49493\pm0{\cdot}1224745(1+0{\cdot}0041666\ldots$$
$$+0{\cdot}000080208+0{\cdot}00000211\ldots),$$

$$y_2 = -0{\cdot}37194,\quad y_3 = -0{\cdot}61792.$$

Only three terms need have been taken in the series to ensure solutions correct to 5 decimal places.

APPENDIX III

ALTERNATIVE RESOLVENT CUBICS FOR FACTORIZING QUARTICS

SEVERAL different methods of factorizing a quartic by solving an auxiliary cubic equation are derived in many textbooks, in various manners. It is interesting that every one of these methods can be obtained, in a particularly simple way, by the scheme given in section 3.1, the only difference consisting in the choice of the auxiliary quantity [(x) in section 3.1], which is always a certain simple combination of the unknown coefficients of two quadratic factors. For instance, *Lagrange's solution*, in a more general form than usually given, may be obtained by introducing the auxiliary quantity:

$$t = H+h \qquad \text{(III.1)}$$

which, in view of (3.6) and then of (3.3) and (3.5), leads to

$$\left.\begin{aligned} H, h &= \frac{t\pm\sqrt{(t^2-4AE)}}{2} \\ G, g &= \frac{B}{2}\pm\frac{Bt-2AD}{2\sqrt{(t^2-4AE)}} \end{aligned}\right\} \qquad \text{(III.2)}$$

and then, substituting in (3.4), to the resolvent cubic

$$t^3-Ct^2+(BD-4AE)t-(AD^2+B^2E-4ACE) = 0. \quad \text{(III.3)}$$

Similarly, *Descartes's solution*, in a considerably generalized form, may be obtained by putting

$$s = G-g \qquad \text{(III.4)}$$

which, in view of (3.3) and then of (3.4) and (3.5), leads to

$$\begin{aligned} G,g &= \frac{B\pm s}{2} \\ H,h &= \frac{s^3+(4AC-B^2)s\pm[Bs^2+(4ABC-B^3-8A^2D)]}{8As} \end{aligned} \qquad \text{(III.5)}$$

and then, on substituting in (3.6), to the resolvent cubic in s^2:

$$\begin{aligned} s^6-(3B^2-8AC)s^4+[(B^2-4AC)(3B^2-4AC) \\ +16A^2(BD-4AE)]s^2-(B^3-4ABC+8A^2D)^2 = 0. \end{aligned} \qquad \text{(III.6)}$$

It can be shown (by some rather awkward transformation) that the well-known but inconvenient *Euler's solution* reduces essentially to the same resolvent (III.6).

Ferrari's solution consists essentially in transforming the quartic so as to present it as a difference of two squares. Thus, in the most general case, (3.1) may be written

$$\left(A\lambda^2+\frac{B\lambda}{2}+\frac{t}{2}\right)^2 = \left(\frac{B^2}{4}-AC+At\right)\lambda^2+\left(\frac{Bt}{2}-AD\right)\lambda+\left(\frac{t^2}{4}-AE\right), \qquad \text{(III.7)}$$

where t may be an arbitrary quantity. The right-hand part becomes a perfect square of a binomial in λ if

$$\left(\frac{Bt}{2}-AD\right)^2 = \left(\frac{B^2}{2}-AC+At\right)(t^2-4AE), \qquad \text{(III.8)}$$

and this relationship, when simplified, becomes identical with (III.3), so that Ferrari's and Lagrange's methods are essentially equivalent.

In the present paper, preference has been given to the resolvent cubic (3.9) for the simple reason that its constant term happens to be Routh's discriminant, and this is obviously advantageous, especially for problems of stability.

TABLE 1

ROOTS OF EQUATION $4y^3-3y = \Delta$

Δ	y_1		y_2		y_3	
0	0·86603	166	0·00000	333	−0·86603	167
0·01	0·86769	165	−0·00333	334	−0·86436	169
0·02	0·86934	164	−0·00667	333	−0·86267	169
0·03	0·87098	164	−0·01000	334	−0·86098	170
0·04	0·87262	162	−0·01334	333	−0·85928	171
0·05	0·87424	162	−0·01667	334	−0·85757	172
0·06	0·87586	160	−0·02001	334	−0·85585	174
0·07	0·87746	160	−0·02335	334	−0·85411	174
0·08	0·87906	159	−0·02669	335	−0·85237	176
0·09	0·88065	158	−0·03004	334	−0·85061	176
0·10	0·88223	158	−0·03338	335	−0·84885	177
0·11	0·88381	156	−0·03673	336	−0·84708	180
0·12	0·88537	156	−0·04009	335	−0·84528	179
0·13	0·88693	155	−0·04344	336	−0·84349	181
0·14	0·88848	154	−0·04680	337	−0·84168	183
0·15	0·89002	153	−0·05017	337	−0·83985	184
0·16	0·89155	153	−0·05354	337	−0·83801	184
0·17	0·89308	152	−0·05691	338	−0·83617	186
0·18	0·89460	151	−0·06029	339	−0·83431	188
0·19	0·89611	150	−0·06368	339	−0·83243	189
0·20	0·89761	150	−0·06707	340	−0·83054	190
0·21	0·89911	148	−0·07047	340	−0·82864	192
0·22	0·90059	149	−0·07387	341	−0·82672	192
0·23	0·90208	147	−0·07728	342	−0·82480	195
0·24	0·90355	147	−0·08070	343	−0·82285	196
0·25	0·90502	146	−0·08413	343	−0·82089	197
0·26	0·90648	145	−0·08756	344	−0·81892	199
0·27	0·90793	145	−0·09100	346	−0·81693	201
0·28	0·90938	144	−0·09446	346	−0·81492	202
0·29	0·91082	144	−0·09792	347	−0·81290	203
0·30	0·91226	143	−0·10139	348	−0·81087	205
0·31	0·91369	142	−0·10487	349	−0·80882	207
0·32	0·91511	141	−0·10836	351	−0·80675	210
0·33	0·91652	141	−0·11187	351	−0·80465	210
0·34	0·91793	141	−0·11538	353	−0·80255	212
0·35	0·91934	139	−0·11891	354	−0·80043	215
0·36	0·92073	139	−0·12245	355	−0·79828	216
0·37	0·92212	139	−0·12600	357	−0·79612	218
0·38	0·92351	138	−0·12957	358	−0·79394	220
0·39	0·92489		−0·13315		−0·79174	

TABLE 1 (contd.)

ROOTS OF EQUATION $4y^3 - 3y = \Delta$

Δ	y_1		y_2		y_3	
0·39	0·92489	137	−0·13315	359	−0·79174	222
0·40	0·92626	137	−9·13674	361	−0·78952	224
0·41	0·92763	136	−0·14035	363	−0·78728	227
0·42	0·92899	136	−0·14398	364	−0·78501	228
0·43	0·93035	135	−0·14762	366	−0·78273	231
0·44	0·93170	134	−0·15128	368	−0·78042	234
0·45	0·93304	135	−0·15496	370	−0·77808	235
0·46	0·93439	133	−0·15866	371	−0·77573	238
0·47	0·93572	133	−0·16237	374	−0·77335	241
0·48	0·93705	132	−0·16611	376	−0·77094	244
0·49	0·93837	132	−0·16987	378	−0·76850	246
0·50	0·93969	132	−0·17365	380	−0·76604	248
0·51	0·94101	131	−0·17745	383	−0·76356	252
0·52	0·94232	130	−0·18128	385	−0·76104	255
0·53	0·94362	130	−0·18513	387	−0·75849	257
0·54	0·94492	129	−0·18900	390	−0·75592	261
0·55	0·94621	129	−0·19290	393	−0·75331	264
0·56	0·94750	128	−0·19683	395	−0·75067	267
0·57	0·94878	128	−0·20079	399	−0·74799	271
0·58	0·95006	128	−0·20478	403	−0·74528	275
0·59	0·95134	127	−0·20881	405	−0·74253	278
0·60	0·95261	126	−0·21286	405	−0·73975	283
0·61	0·95387	126	−0·21695	412	−0·73692	286
0·62	0·95513	126	−0·22107	417	−0·73406	291
0·63	0·95639	125	−0·22524	420	−0·73115	295
0·64	0·95764	125	−0·22944	424	−0·72820	299
0·65	0·95889	124	−0·23368	429	−0·72521	305
0·66	0·96013	124	−0·23797	433	−0·72216	309
0·67	0·96137	123	−0·24230	438	−0·71907	315
0·68	0·96260	123	−0·24668	443	−0·71592	320
0·69	0·96383	123	−0·25111	449	−0·71272	326
0·70	0·96506	122	−0·25560	454	−0·70946	332
0·71	0·96628	122	−0·26014	460	−0·70614	338
0·72	0·96750	121	−0·26474	466	−0·70276	345
0·73	0·96871	121	−0·26940	474	−0·69931	353
0·74	0·96992	120	−0·27414	480	−0·69578	360
0·75	0·97112	120	−0·27894	488	−0·69218	368
0·76	0·97232	120	−0·28382	496	−0·68850	376
0·77	0·97352	119	−0·28878	504	−0·68474	385
0·78	0·97471		−0·29382		−0·68089	

TABLE 1 (contd.)

ROOTS OF EQUATION $4y^3 - 3y = \Delta$

Δ	y_1		y_2		y_3	
0·78	0·97471	119	−0·29382	514	−0·68089	395
0·79	0·97590	118	−0·29896	524	−0·67694	406
0·80	0·97708	118	−0·30420	535	−0·67288	417
0·81	0·97826	118	−0·30955	546	−0·66871	428
0·82	0·97944	117	−0·31501	560	−0·66443	443
0·83	0·98061	117	−0·32061	573	−0·66000	456
0·84	0·98178	117	−0·32634	588	−0·65544	471
0·85	0·98295	116	−0·33222	606	−0·65073	490
0·86	0·98411	116	−0·33828	625	−0·64583	509
0·87	0·98527	115	−0·34453	645	−0·64074	530
0·88	0·98642	115	−0·35098	670	−0·63544	555
0·89	0·98757	115	−0·35768	697	−0·62989	582
0·90	0·98872	114	−0·36465	729	−0·62407	615
0·91	0·98986	114	−0·37194	766	−0·61792	652
0·92	0·99100	114	−0·37960	810	−0·61140	696
0·93	0·99214	113	−0·38770	865	−0·60444	752
0·94	0·99327	113	−0·39635	935	−0·59692	822
0·95	0·99440	113	−0·40570	1027	−0·58870	914
0·96	0·99553	112	−0·41597	1184	−0·57956	1042
0·97	0·99665	112	−0·42751	1358	−0·56914	1246
0·98	0·99777	112	−0·44109	1752	−0·55668	1640
0·99	0·99889	111	−0·45861	4139	−0·54028	4028
1·00	1·00000		−0·50000		−0·50000	

Δ	y_1		y_2, y_3	
1·00	1·00000	111	−0·50000 ± 0·00000i	4080
1·01	1·00111	111	−0·50056 ± 0·04080i	1688
1·02	1·00222	110	−0·50111 ± 0·05768i	1294
1·03	1·00332	110	−0·50166 ± 0·07062i	1089
1·04	1·00442	110	−0·50221 ± 0·08151i	957
1·05	1·00552	109	−0·50276 ± 0·09108i	865
1·06	1·00661	109	−0·50331 ± 0·09973i	794
1·07	1·00770	109	−0·50385 ± 0·10767i	738
1·08	1·00879	108	−0·50439 ± 0·11505i	692
1·09	1·00987	108	−0·50493 ± 0·12197i	655
1·10	1·01095	108	−0·50548 ± 0·12852i	621
1·11	1·01203	107	−0·50601 ± 0·13473i	592
1·12	1·01310	108	−0·50655 ± 0·14065i	568
1·13	1·01418		−0·50709 ± 0·14633i	

TABLE 1 (contd.)

ROOTS OF EQUATION $4y^3 - 3y = \Delta$

Δ	y_1		y_2, y_3	
1·13	1·01418	106	−0·50709 ± 0·14633i	546
1·14	1·01524	107	−0·50762 ± 0·15179i	526
1·15	1·01631	106	−0·50816 ± 0·15705i	508
1·16	1·01737	106	−0·50869 ± 0·16213i	492
1·17	1·01843	106	−0·50922 ± 0·16705i	476
1·18	1·01949	105	−0·50975 ± 0·17181i	463
1·19	1·02054	106	−0·51027 ± 0·17644i	451
1·20	1·02160		−0·51080 ± 0·18095i	
1·2	1·02160	1036	−0·51080 ± 0·18095i	3973
1·3	1·03196	1009	−0·51598 ± 0·22068i	3311
1·4	1·04205	985	−0·52103 ± 0·25379i	2883
1·5	1·05190	962	−0·52595 ± 0·28262i	2579
1·6	1·06152	939	−0·53076 ± 0·30841i	2347
1·7	1·07091	919	−0·53546 ± 0·33188i	2163
1·8	1·08010	900	−0·54005 ± 0·35351i	2013
1·9	1·08910	881	−0·54455 ± 0·37364i	1886
2·0	1·09791	864	−0·54896 ± 0·39250i	1779
2·1	1·10655	847	−0·55327 ± 0·41029i	1685
2·2	1·11502	831	−0·55751 ± 0·42714i	1603
2·3	1·12333	816	−0·56167 ± 0·44317i	1531
2·4	1·13149	802	−0·56575 ± 0·45848i	1466
2·5	1·13951	788	−0·56975 ± 0·47314i	1407
2·6	1·14739	775	−0·57369 ± 0·48721i	1355
2·7	1·15514	762	−0·57757 ± 0·50076i	1306
2·8	1·16276	750	−0·58138 ± 0·51382i	1262
2·9	1·17026	739	−0·58513 ± 0·52644i	1221
3·0	1·17765	728	−0·58883 ± 0·53865i	1184
3·1	1·18493	716	−0·59246 ± 0·55049i	1149
3·2	1·19209	707	−0·59605 ± 0·56198i	1116
3·3	1·19916	696	−0·59958 ± 0·57314i	1086
3·4	1·20612	687	−0·60306 ± 0·58400i	1057
3·5	1·21299	678	−0·60650 ± 0·59457i	1031
3·6	1·21977	669	−0·60989 ± 0·60488i	1006
3·7	1·22646	660	−0·61323 ± 0·61494i	982
3·8	1·23306	652	−0·61653 ± 0·62476i	960
3·9	1·23958	644	−0·61979 ± 0·63436i	939
4·0	1·24602	636	−0·62301 ± 0·64375i	919
4·1	1·25238	628	−0·62619 ± 0·65294i	900
4·2	1·25866		−0·62933 ± 0·66194i	

TABLE 1 (contd.)

ROOTS OF EQUATION $4y^3 - 3y = \Delta$

Δ	y_1		y_2, y_3	
4·2	1·25866	621	−0·62933 ± 0·66194i	882
4·3	1·26487	614	−0·63243 ± 0·67076i	865
4·4	1·27101	607	−0·63550 ± 0·67941i	848
4·5	1·27708	600	−0·63854 ± 0·68789i	832
4·6	1·28308	593	−0·64154 ± 0·69621i	818
4·7	1·28901	587	−0·64451 ± 0·70439i	803
4·8	1·29488	581	−0·64744 ± 0·71242i	790
4·9	1·30069	575	−0·65035 ± 0·72032i	776
5·0	1·30644	570	−0·65322 ± 0·72808i	763
5·1	1·31214	563	−0·65607 ± 0·73571i	752
5·2	1·31777	558	−0·65888 ± 0·74323i	740
5·3	1·32335	552	−0·66167 ± 0·75063i	728
5·4	1·32887	547	−0·66444 ± 0·75791i	717
5·5	1·33434	542	−0·66717 ± 0·76508i	707
5·6	1·33976	537	−0·66988 ± 0·77215i	697
5·7	1·34513	532	−0·67257 ± 0·77912i	688
5·8	1·35045	527	−0·67523 ± 0·78600i	677
5·9	1·35572	523	−9·67786 ± 0·79277i	669
6·0	1·36095	517	−0·68047 ± 0·79946i	659
6·1	1·36612	514	−0·68206 ± 0·80605i	652
6·2	1·37126	509	−0·68563 ± 0·81257i	643
6·3	1·37635	504	−0·68817 ± 0·81900i	634
6·4	1·38139	501	−0·69070 ± 0·82534i	627
6·5	1·38640	496	−0·69320 ± 0·83161i	619
6·6	1·39136	492	−0·69568 ± 0·83780i	613
6·7	1·39628	489	−0·69814 ± 0·84393i	604
6·8	1·40117	484	−0·70058 ± 0·84997i	598
6·9	1·40601	481	−0·70301 ± 0·85595i	591
7·0	1·41082	477	−0·70541 ± 0·86186i	585
7·1	1·41559	473	−0·70780 ± 0·86771i	578
7·2	1·42032	470	−0·71016 ± 0·87349i	572
7·3	1·42502	466	−0·71251 ± 0·87921i	566
7·4	1·42968	463	−0·71484 ± 0·88487i	560
7·5	1·43431	460	−0·71716 ± 0·89047i	555
7·6	1·43891	456	−0·71945 ± 0·89602i	548
7·7	1·44347	453	−0·72173 ± 0·90150i	544
7·8	1·44800	449	−0·72400 ± 0·90694i	537
7·9	1·45249	447	−0·72625 ± 0·91231i	532
8·0	1·45696		−0·72848 ± 0·91763i	

TABLE 1 (contd.)

ROOTS OF EQUATION $4y^3-3y=\Delta$

Δ	y_1		y_2, y_3	
8.0	1·45696	443	−0·72848 ± 0·91763i	527
8·1	1·46139	441	−0·73070 ± 0·92290i	523
8·2	1·46580	437	−0·73290 ± 0·92813i	517
8·3	1·47017	435	−0·73509 ± 0·93330i	513
8·4	1·47452	431	−9·73726 ± 0·93843i	507
8·5	1·47883	429	−0·73942 ± 0·94351i	503
8·6	1·48312	426	−0·74156 ± 0·94854i	499
8·7	1·48738	423	−0·74369 ± 0·95353i	495
8·8	1·49161	421	−0·74581 ± 0·95848i	491
8·9	1·49582	418	−0·74791 ± 0·96339i	486
9·0	1·50000	415	−0·75000 ± 0·96825i	481
9·1	1·50415	413	−0·75208 ± 0·97306i	479
9·2	1·50828	411	−0·75414 ± 0·97785i	474
8·3	1·51239	407	−0·75619 ± 0·98259i	470
9·4	1·51646	406	−0·75823 ± 0·98729i	467
9·5	1·52052	403	−0·76026 ± 0·99196i	463
9·6	1·52455	400	−0·76227 ± 0·99659i	459
9·7	1·52855	398	−0·76428 ± 1·00118i	455
9·8	1·53253	396	−0·76627 ± 1·00573i	452
9·9	1·53649	394	−0·76825 ± 1·01025i	449
10	1·54043	3819	−0·77021 ± 1·01474i	4310
11	1·57862	3623	−0·78931 ± 1·05784i	4025
12	1·61485	3453	−0·80743 ± 1·09809i	3784
13	1·64938	3300	−0·82469 ± 1·13593i	3574
14	1·68238	3163	−0·84119 ± 1·17167i	3389
15	1·71401	3041	−0·85701 ± 1·20556i	3228
16	1·74442	2930	−0·87221 ± 1·23784i	3085
17	1·77372	2828	−0·88686 ± 1·26869i	2954
18	1·80200	2736	−0·90100 ± 1·29823i	2839
19	1·82936	2649	−0·91468 ± 1·32662i	2731
20	1·85585	2570	−0·92793 ± 1·35393i	2635
21	1·88155	2497	−0·94078 ± 1·38028i	2546
22	1·90652	2429	−0·95326 ± 1·40574i	2465
23	1·93081	2365	−0·96540 ± 1·43039i	2389
24	1·95446	2304	−0·97723 ± 1·45428i	2319
25	1·97751	2249	−0·98875 ± 1·47747i	2253
26	2·00000	2196	−1·00000 ± 1·50000i	2192
27	2·02196	2147	−1·01098 ± 1·52192i	2136
28	2·04343	2100	−1·02172 ± 1·54328i	2082
29	2·06443		−1·03221 ± 1·56410i	

TABLE 1 (contd.)

ROOTS OF EQUATION $4y^3-3y=\Delta$

Δ	y_1		y_2, y_3	
29	2·06443	2055	−1·03221 ± 1·56410i	2031
30	2·08498	2014	−1·04249 ± 1·58441i	1984
31	2·10512	1973	−1·05256 ± 1·60425i	1940
32	2·12485	1935	−1·06242 ± 1·62365i	1897
33	2·14420	1899	−1·07210 ± 1·64262i	1857
34	2·16319	1864	−1·08160 ± 1·66119i	1818
35	2·18183	1832	−1·09092 ± 1·67937i	1783
36	2·20015	1799	−1·10007 ± 1·69720i	1748
37	2·21814	1770	−1·10907 ± 1·71468i	1715
38	2·23584	1740	−1·11792 ± 1·73183i	1683
39	2·25324	1713	−1·12662 ± 1·74866i	1654
40	2·27037	1686	−1·13518 ± 1·76520i	1625
41	2·28723	1660	−1·14361 ± 1·78145i	1597
42	2·30383	1635	−1·15191 ± 1·79742i	1571
43	2·32018	1612	−1·16009 ± 1·81313i	1546
44	2·33630	1589	−1·16815 ± 1·82859i	1521
45	2·35219	1566	−1·17609 ± 1·84380i	1497
46	2·36785	1545	−1·18393 ± 1·85877i	1476
47	2·38330	1525	−1·19165 ± 1·87353i	1453
48	2·39855	1504	−1·19927 ± 1·88806i	1433
49	2·41359	1485	−1·20680 ± 1·90239i	1412
50	2·42844	1467	−1·21422 ± 1·91651i	1392
51	2·44311	1448	−1·22155 ± 1·93043i	1374
52	2·45759	1430	−1·22879 ± 1·94417i	1356
53	2·47189	1414	−1·23595 ± 1·95773i	1338
54	2·48603	1397	−1·24301 ± 1·97111i	1320
55	2·50000	1381	−1·25000 ± 1·98431i	1305
56	2·51381	1365	−1·25690 ± 1·99736i	1288
57	2·52746	1350	−1·26373 ± 2·01024i	1272
58	2·54096	1336	−1·27048 ± 2·02296i	1258
59	2·55432	1321	−1·27716 ± 2·03554i	1242
60	2·56753	1307	−1·28376 ± 2·04796i	1229
61	2·58060	1293	−1·29030 ± 2·06025i	1214
62	2·59353	1280	−1·29677 ± 2·07239i	1201
63	2·60633	1268	−1·30317 ± 2·08440i	1188
64	2·61901	1254	−1·30950 ± 2·09628i	1175
65	2·63155i	1243	−1·31578 ± 2·10803i	1163
66	2·64398	1230	−1·32199 ± 2·11966i	1150
67	2·65628	1218	−1·32814 ± 2·13116i	1139
68	2·66846		−1·33423 ± 2·14255i	

Table 1 (contd.)

Roots of equation $4y^3 - 3y = \Delta$

Δ	y_1		y_2, y_3	
68	2·66846	1208	−1·33423 ± 2·14255i	1127
69	2·68054	1196	−1·34027 ± 2·15382i	1117
70	2·69250	1185	−1·34625 ± 2·16499i	1105
71	2·70435	1175	−1·35217 ± 2·17604i	1094
72	2·71610	1163	−1·35805 ± 2·18698i	1084
73	2·72773	1154	−1·36387 ± 2·19782i	1073
74	2·73927	1144	−1·36964 ± 2·20855i	1064
75	2·75071	1134	−1·37536 ± 2·21919i	1054
76	2·76205	1125	−1·38103 ± 2·22973i	1045
77	2·77330	1115	−1·38665 ± 2·24018i	1035
78	2·78445	1106	−1·39223 ± 2·25053i	1026
79	2·79551	1097	−1·39776 ± 2·26079i	1017
80	2·80648	1089	−1·40324 ± 2·27096i	1009
81	2·81737	1079	−1·40868 ± 2·28105i	999
82	2·82816	1072	−1·41408 ± 2·29104i	992
83	2·83888	1063	−1·41944 ± 2·30096i	984
84	2·84951	1055	−1·42475 ± 2·31080i	975
85	2·86006	1047	−1·43003 ± 2·32055i	968
86	2·87053	1039	−1·43526 ± 2·33023i	959
87	2·88092	1031	−1·44046 ± 2·33982i	952
88	2·89123	1024	−1·44562 ± 2·34934i	945
89	2·90147	1016	−1·45074 ± 2·35879i	937
90	2·91163	1010	−1·45582 ± 2·36816i	931
91	2·92173	1001	−1·46086 ± 2·37747i	923
92	2·93174	996	−1·46587 ± 2·38670i	917
93	2·94170	988	−1·47085 ± 2·39587i	910
94	2·95158	982	−1·47579 ± 2·40497i	903
95	2·96140	974	−1·48070 ± 2·41400i	897
96	2·97114	969	−1·48557 ± 2·42297i	890
97	2·98083	961	−1·49041 ± 2·43187i	884
98	2·99044	956	−1·49522 ± 2·44071i	878
99	3·00000	949	−1·50000 ± 2·44949i	872
100	3·00949	943	−1·50475 ± 2·45821i	866
101	3·01892	938	−1·50946 ± 2·46687i	860
102	3·02830	931	−1·51415 ± 2·47547i	854
103	3·03761	925	−1·51880 ± 2·48401i	848
104	3·04686	920	−1·52343 ± 2·49249i	843
105	3·05606	914	−1·52803 ± 2·50092i	838
106	3·06520	908	−1·53260 ± 2·50930i	832
107	3·07428		−1·53714 ± 2·51762i	

Table 1 (contd.)

Roots of equation $4y^3 - 3y = \Delta$

Δ	y_1		y_2, y_3	
107	3·07428	903	−1·53714 ± 2·51762i	827
108	3·08331	898	−1·54166 ± 2·52589i	821
109	3·09229	892	−1·54614 ± 2·53410i	817
110	3·10121	887	−1·55061 ± 2·54227i	811
111	3·11008	882	−1·55504 ± 2·55038i	807
112	3·11890	877	−1·55945 ± 2·55845i	801
113	3·12767	871	−1·56383 ± 2·56646i	797
114	3·13638	867	−1·56819 ± 2·57443i	792
115	3·14505	862	−1·57253 ± 2·58235i	787
116	3·15367	857	−1·57684 ± 2·59022i	782
117	3·16224	852	−1·58112 ± 2·59804i	778
118	3·17076	848	−1·58538 ± 2·60582i	774
119	3·17924	843	−1·58962 ± 2·61356i	769
120	3·18767	839	−1·59384 ± 2·62125i	765
121	3·19606	834	−1·59803 ± 2·62890i	760
122	3·20440	829	−1·60220 ± 2·63650i	756
123	3·21269	826	−1·60635 ± 2·64406i	752
124	3·22095	821	−1·61047 ± 2·65158i	748
125	3·22916	816	−1·61458 ± 2·65906i	744
126	3·23732	813	−1·61866 ± 2·66650i	739
127	3·24545	808	−1·62272 ± 2·67389i	736
128	3·25353	804	−1·62677 ± 2·68125i	732
129	3·26157	801	−1·63079 ± 2·68857i	728
130	3·26958	796	−1·63479 ± 2·69585i	724
131	3·27754	792	−1·63877 ± 2·70309i	720
132	3·28546	788	−1·64273 ± 2·71029i	717
133	3·29334	785	−1·64667 ± 2·71746i	713
134	3·30119	781	−1·65060 ± 2·72459i	710
135	3·30900	777	−1·65450 ± 2·73169i	706
136	3·31677	773	−1·65839 ± 2·73875i	702
137	3·32450	770	−1·66225 ± 2·74577i	699
138	3·33220	766	−1·66610 ± 2·75276i	695
139	3·33986	762	−1·66998 ± 2·75971i	692
140	3·34748	759	−1·67374 ± 2·76663i	688
141	3·35507	755	−1·67754 ± 2·77351i	685
142	3·36262	752	−1·68131 ± 2·78036i	682
143	3·37014	749	−1·68507 ± 2·78718i	679
144	3·37763	745	−1·68881 ± 2·79397i	676
145	3·38508	742	−1·69254 ± 2·80073i	672
146	3·39250		−1·69625 ± 2·80745i	

TABLE 1 (concluded)

ROOTS OF EQUATION $4y^3 - 3y = \Delta$

Δ	y_1		y_2, y_3	
146	3·39250	738	−1·69625±2·80745i	669
147	3·39988	735	−1·69994±2·81414i	666
148	3·40723	732	−1·70362±2·82080i	663
149	3·41455	729	−1·70728±2·82743i	661
150	3·42184	726	−1·71092±2·83404i	657
151	3·42910	723	−1·71455±2·84061i	654
152	3·43633	719	−1·71816±2·84715i	651
153	3·44352	716	−1·72176±2·85366i	648
154	3·45068	714	−1·72534±2·86014i	646
155	3·45782	710	−1·72891±2·86660i	642
156	3·46492	707	−1·73246±2·87302i	640
157	3·47199	705	−1·73600±2·87942i	637
158	3·47904	701	−1·73952±2·88579i	634
159	3·48605	699	−1·74303±2·89213i	632
160	3·49304	696	−1·74602±2·89845i	629
161	3·50000	693	−1·75000±2·90474i	626
162	3·50693	690	−1·75347±2·91100i	624
163	3·51383	688	−1·75692±2·91724i	621
164	3·52071	685	−1·76035±2·92345i	618
165	3·52756	682	−1·76378±2·92963i	616
166	3·53438	679	−1·76719±2·93579i	613
167	3·54117	677	−1·77059±2·94192i	611
168	3·54794	674	−1·77397±2·94803i	609
169	3·55468	671	−1·77734±2·95412i	606
170	3·56139		−1·78070±2·96018i	

TABLE 2

ROOTS OF EQUATION $4z^3 + 3z = \Delta$

Δ	z_1		z_2, z_3	
0	0·00000	333	−0·00000±0·86603i	0
0·01	0·00333	334	−0·00167±0·86603i	1
0·02	0·00667	333	−0·00333±0·86604i	3
0·03	0·01000	333	−0·00500±0·86607i	3
0·04	0·01333	333	−0·00667±0·86610i	4
0·05	0·01666	333	−0·00833±0·86614i	6
0·06	0·01999	333	−0·00999±0·86620i	6
0·07	0·02332	332	−0·01166±0·86626i	7
0·08	0·02664	332	−0·01332±0·86633i	8
0·09	0·02996	332	−0·01498±0·86641i	9
0·10	0·03328	332	−0·01664±0·86650i	11
0·11	0·03660	332	−0·01830±0·86661i	11
0·12	0·03992	331	−0·01996±0·86672i	12
0·13	0·04323	330	−0·02161±0·86684i	12
0·14	0·04653	331	−0·02327±0·86696i	14
0·15	0·04984	329	−0·02492±0·86710i	14
0·16	0·05313	330	−0·02657±0·86724i	16
0·17	0·05643	329	−0·02821±0·86740i	17
0·18	0·05972	328	−0·02986±0·86757i	17
0·19	0·06300	328	−0·03150±0·86774i	19
0·20	0·06628	327	−0·03314±0·86793i	19
0·21	0·06955	327	−0·03478±0·86812i	20
0·22	0·07282	326	−0·03641±9·86832i	21
0·23	0·07608	325	−0·03804±0·86853i	21
0·24	0·07933	325	−0·03967±0·86874i	23
0·25	0·08258	324	−0·04129±0·86897i	24
0·26	0·08582	324	−0·04291±0·86921i	24
0·27	0·08906	323	−0·04453±0·86945i	25
0·28	0·09229	321	−0·04614±0·86970i	27
0·29	0·09550	322	−0·04775±0·86997i	27
0·30	0·09872	320	−0·04936±9·87024i	27
0·31	0·10192	320	−0·05096±0·87051i	29
0·32	0·10512	319	−0·05256±0·87080i	29
0·33	0·10831	318	−0·05415±0·87109i	30
0·34	0·11149	317	−0·05574±0·87139i	31
0·35	0·11466	316	−0·05733±0·87170i	32
0·36	0·11782	315	−0·05891±0·87202i	32
0·37	0·12097	315	−0·06049±0·87234i	33
0·38	0·12412	313	−0·06206±0·87267i	34
0·39	0·12725		−0·06363±0·87301i	

TABLE 2 (contd.)

ROOTS OF EQUATION $4z^3+3z = \Delta$

Δ	z_1		z_2, z_3	
0·39	0·12725	313	−0·06363 ± 0·87301i	35
0·40	0·13038	312	−0·06519 ± 0·87336i	35
0·41	0·13350	310	−0·06675 ± 0·87371i	36
0·42	0·13660	310	−0·06830 ± 0·87407i	37
0·43	0·13970	308	−0·06985 ± 0·87444i	37
0·44	0·14278	308	−0·07139 ± 0·87481i	38
0·45	0·14586	307	−0·07293 ± 0·87519i	39
0·46	0·14893	306	−0·07446 ± 0·87558i	39
0·47	0·15199	304	−0·07599 ± 0·87597i	40
0·48	0·15503	304	−0·07752 ± 0·87637i	41
0·49	0·15807	302	−0·07903 ± 0·87678i	41
0·50	0·16109	302	−0·08055 ± 0·87719i	42
0·51	0·16411	300	−0·08205 ± 0·87761i	42
0·52	0·16711	299	−0·08356 ± 0·87803i	43
0·53	0·17010	299	−0·08505 ± 0·87846i	44
0·54	0·17309	297	−0·08654 ± 0·87890i	44
0·55	0·17606	296	−0·08803 ± 0·87934i	46
0·56	0·17902	295	−0·08951 ± 0·87980i	45
0·57	0·18197	293	−0·09098 ± 0·88025i	46
0·58	0·18490	293	−0·09245 ± 0·88071i	46
0·59	0·18783	292	−0·09397 ± 0·88117i	47
0·60	0·19075	290	−0·09537 ± 0·88164i	47
0·61	0·19365	289	−0·09683 ± 0·88211i	48
0·62	0·19654	288	−0·09827 ± 0·88259i	49
0·63	0·19942	288	−0·09971 ± 0·88308i	49
0·64	0·20230	285	−0·10115 ± 0·88357i	49
0·65	0·20515	285	−0·10258 ± 0·88406i	50
0·66	0·20800	284	−0·10400 ± 0·88456i	51
0·67	0·21084	282	−0·10542 ± 0·88507i	50
0·68	0·21366	282	−0·10683 ± 0·88557i	51
0·69	0·21648	280	−0·10824 ± 0·88608i	52
0·70	0·21928	278	−0·10964 ± 0·88660i	52
0·71	0·22206	278	−9·11103 ± 0·88712i	53
0·72	0·22484	277	−0·11242 ± 0·88765i	53
0·73	0·22761	276	−0·11381 ± 0·88818i	53
0·74	0·23037	274	−0·11518 ± 0·88871i	53
0·75	0·23311	273	−0·11656 ± 0·88924i	54
0·76	0·23584	272	−0·11792 ± 0·88978i	55
0·77	0·23856	271	−0·11928 ± 0·89033i	55
0·78	0·24127		−0·12064 ± 0·89088i	

TABLE 2 (contd.)

ROOTS OF EQUATION $4z^3+3z = \Delta$

Δ	z_1		z_2, z_3	
0·78	0·24127	270	−0·12064 ± 0·89088i	55
0·79	0·24397	269	−0·12199 ± 0·89143i	55
0·80	0·24666	267	−0·12333 ± 0·89198i	56
0·81	0·24933	267	−0·12467 ± 0·89254i	56
0·82	0·25200	265	−0·12600 ± 0·89310i	56
0·83	0·25465	264	−0·12732 ± 0·89366i	57
0·84	0·25729	263	−0·12870 ± 0·89423i	57
0·85	0·25992	262	−0·12996 ± 0·89480i	57
0·86	0·26254	261	−0·13127 ± 0·89537i	58
0·87	0·26515	259	−0·13257 ± 0·89595i	57
0·88	0·26774	259	−0·13387 ± 0·89652i	59
0·89	0·27033	257	−0·13516 ± 0·89711i	58
0·90	0·27290	256	−0·13645 ± 0·89769i	59
0·91	0·27546	256	−0·13773 ± 0·89828i	59
0·92	0·27802	254	−0·13901 ± 0·89887i	59
0·93	0·28056	252	−0·14028 ± 0·89946i	60
0·94	0·28308	252	−0·14154 ± 0·90006i	59
0·95	0·28560	251	−0·14280 ± 0·90065i	60
0·96	0·28811	250	−0·14406 ± 0·90125i	60
0·97	0·29061	249	−0·14530 ± 0·90185i	61
0·98	0·29310	247	−0·14655 ± 0·90246i	60
0·99	0·29557	246	−0·14779 ± 0·90306i	61
1·0	0·29803	2407	−0·14902 ± 0·90367i	617
1·1	0·32210	2307	−0·16105 ± 0·90984i	632
1·2	0·34517	2211	−0·17258 ± 0·91616i	643
1·3	0·36728	2121	−0·18364 ± 0·92259i	649
1·4	0·38849	2038	−0·19424 ± 0·92908i	654
1·5	0·40887	1959	−0·20443 ± 0·93562i	655
1·6	0·42846	1886	−0·21423 ± 0·94217i	655
1·7	0·44732	1819	−0·22366 ± 0·94872i	654
1·8	0·46551	1754	−0·23275 ± 0·95526i	651
1·9	0·48305	1695	−0·24152 ± 0·96177i	648
2·0	0·50000	1640	−0·25000 ± 0·96825i	643
2·1	0·51640	1587	−0·25820 ± 0·97468i	640
2·2	0·53227	1539	−0·26613 ± 0·98108i	631
2·3	0·54766	1493	−0·27383 ± 0·98739i	628
2·4	0·56259	1449	−0·28129 ± 0·99367i	621
2·5	0·57708	1410	−0·28854 ± 0·99988i	616
2·6	0·59118		−0·29559 ± 1·00604i	

TABLE 2 (contd.)

ROOTS OF EQUATION $4z^3+3z = \Delta$

Δ	z_1		z_2, z_3	
2·6	0·59118	1372	−0·29559 ± 1·00604i	610
2·7	0·60490	1335	−0·30245 ± 1·01214i	603
2·8	0·61825	1301	−0·30912 ± 1·01817i	597
2·9	0·63126	1270	−0·31563 ± 1·02414i	591
3·0	0·64396	1238	−0·32198 ± 1·03005i	585
3·1	0·65634	1210	−0·32817 ± 1·03590i	579
3·2	0·66844	1183	−0·33422 ± 1·04169i	572
3·3	0·68027	1156	−0·34013 ± 1·04741i	566
3·4	0·69183	1131	−0·34596 ± 1·05307i	561
3·5	0·70314	1108	−0·35157 ± 1·05868i	555
3·6	0·71422	1085	−0·35711 ± 1·06423i	549
3·7	0·72507	1064	−0·36254 ± 1·06972i	544
3·8	0·73571	1043	−0·36786 ± 1·07516i	537
3·9	0·74614	1023	−0·37307 ± 1·08053i	533
4·0	0·75637	1005	−0·37819 ± 1·08586i	526
4·1	0·76642	986	−0·38321 ± 1·09112i	522
4·2	0·77628	969	−0·38814 ± 1·09634i	516
4·3	0·78597	952	−0·39298 ± 1·10150i	511
4·4	0·79549	936	−0·39774 ± 1·10661i	507
4·5	0·80485	920	−0·40242 ± 1·11168i	502
4·6	0·81405	906	−0·40703 ± 1·11670i	497
4·7	0·82311	891	−0·41156 ± 1·12167i	491
4·8	0·83202	878	−0·41601 ± 1·12658i	488
4·9	0·84080	864	−0·42040 ± 1·13146i	483
5·0	0·84944	852	−0·42472 ± 1·13629i	479
5·1	0·85796	839	−0·42898 ± 1·14108i	474
5·2	0·86635	827	−0·43317 ± 1·14582i	471
5·3	0·87462	815	−0·43731 ± 1·15053i	465
5·4	0·88277	804	−0·44138 ± 1·15518i	462
5·5	0·89081	793	−0·44540 ± 1·15980i	459
5.6	0·89874	783	−0·44937 ± 1·16439i	454
5·7	0·90657	772	−0·45328 ± 1·16893i	451
5·8	0·91429	763	−0·45715 ± 1·17344i	446
5·9	0·92192	752	−0·46096 ± 1·17790i	443
6.0	0·92944	744	−0·46472 ± 1·18233i	439
6·1	0·93688	734	−0·46844 ± 1·18672i	436
6·2	0·94422	726	−0·47211 ± 1·19108i	432
6·3	0·95148	717	−0·47574 ± 1·19540i	430
6·4	0·95865	709	−0·47933 ± 1·19970i	425
6·5	0·96574		−0·48287 ± 1·20395i	

TABLE 2 (contd.)

ROOTS OF EQUATION $4z^3+3z = \Delta$

Δ	z_1		z_2, z_3	
6·5	0·96574	700	−0·48287 ± 1·20395i	422
6·6	0·97274	693	−0·48637 ± 1·20817i	420
6·7	0·97967	685	−0·48984 ± 1·21237i	415
6·8	0·98652	678	−0·49326 ± 1·21652i	413
6·9	0·99330	670	−0·49665 ± 1·22065i	409
7·0	1·00000	663	−0·50000 ± 1·22474i	408
7·1	1·00663	656	−0·50332 ± 1·22882i	403
7·2	1·01319	650	−0·50660 ± 1·23285i	401
7·3	1·01969	643	−0·50989 ± 1·23686i	398
7·4	1·02612	636	−0·51306 ± 1·24084i	395
7·5	1·03248	630	−0·51624 ± 1·24479i	392
7·6	1·03878	624	−0·51939 ± 1·24871i	391
7·7	1·04502	618	−0·52251 ± 1·25262i	387
7·8	1·05120	612	−0·52560 ± 1·25649i	385
7·9	1·05732	606	−0·52866 ± 1·26034i	381
8·0	1·06338	601	−0·53169 ± 1·26415i	381
8·1	1·06939	595	−0·53470 ± 1·26796i	376
8·2	1·07534	590	−0·53767 ± 1·27172i	374
8·3	1·08124	585	−0·54062 ± 1·27546i	372
8·4	1·08709	579	−0·54354 ± 1·27918i	370
8·5	1·09288	575	−0·54644 ± 1·28288i	368
8·6	1·09863	569	−0·54931 ± 1·28656i	365
8·7	1·10432	565	−0·55216 ± 1·29021i	364
8·8	1·10997	560	−0·55499 ± 1·29385i	359
8·9	1·11557	555	−0·55778 ± 1·29744i	359
9·0	1·12112	551	−0·56056 ± 1·30103i	357
9·1	1·12663	546	−0·56332 ± 1·30460i	354
9·2	1·13209	542	−0·56605 ± 1·30814i	352
9·3	1·13751	538	−0·56876 ± 1·31166i	349
9·4	1·14289	533	−0·57144 ± 1·31515i	349
9·5	1·14822	530	−0·57411 ± 1·31864i	346
9·6	1·15352	525	−0·57676 ± 1·32210i	344
9·7	1·15877	521	−0·57938 ± 1·32554i	342
9·8	1·16398	517	−0·58199 ± 1·32896i	341
9·9	1·16915	514	−0·58458 ± 1·33237i	337
10	1·17429	4937	−0·58714 ± 1·33574i	3284
11	1·22366	4618	−0·61183 ± 1·36858i	3120
12	1·26984	4344	−0·63492 ± 1·39978i	2974
13	1·31328	4107	−0·65664 ± 1·42952i	2846
14	1·35435		−0·67718 ± 1·45798i	

TABLE 2 (contd.)

ROOTS OF EQUATION $4z^3-3z = \Delta$

Δ	z_1		z_2, z_3	
14	1·35435	3899	$-0·67718 \pm 1·45798i$	2730
15	1·39334	3714	$-0·69667 \pm 1·48528i$	2624
16	1·43048	3550	$-0·71524 \pm 1·51152i$	2530
17	1·46598	3402	$-0·73299 \pm 1·53682i$	2442
18	1·50000	3269	$-0·75000 \pm 1·56124i$	2364
19	1·53269	3147	$-0·76634 \pm 1·58488i$	2290
20	1·56416	3037	$-0·78208 \pm 1·60778i$	2221
21	1·59453	2934	$-0·79726 \pm 1·62999i$	2160
22	1·62387	2841	$-0·81199 \pm 1·65159i$	2099
23	1·65228	2754	$-0·82614 \pm 1·67258i$	2045
24	1·67982	2674	$-0·83991 \pm 1·69303i$	1994
25	1·70656	2598	$-0·85328 \pm 1·71297i$	1945
26	1·73254	2529	$-0·86627 \pm 1·73242i$	1899
27	1·75783	2463	$-0·87891 \pm 1·75141i$	1858
28	1·78246	2401	$-0·89123 \pm 1·76999i$	1817
29	1·80647	2344	$-0·90324 \pm 1·78816i$	1777
30	1·82991	2289	$-0·91495 \pm 1·80593i$	1743
31	1·85280	2237	$-0·92640 \pm 1·82336i$	1708
32	1·87517	2189	$-0·93759 \pm 1·84044i$	1674
33	1·89706	2144	$-0·94853 \pm 1·85718i$	1643
34	1·91848	2099	$-0·95924 \pm 1·87361i$	1613
35	1·93947	2057	$-0·96973 \pm 1·88974i$	1586
36	1·96004	2017	$-0·98002 \pm 1·90560i$	1557
37	1·98021	1979	$-0·99010 \pm 1·92117i$	1532
38	2·00000	1943	$-1·00000 \pm 1·93649i$	1506
39	2·01943	1908	$-1·00976 \pm 1·95155i$	1484
40	2·03851	1876	$-1·01926 \pm 1·96639i$	1458
41	2·05727	1843	$-1·02863 \pm 1·98097i$	1437
42	2·07570	1813	$-1·03785 \pm 1·99534i$	1417
43	2·09383	1784	$-1·04692 \pm 2·00951i$	1394
44	2·11167	1756	$-1·05583 \pm 2·02345i$	1375
45	2·12923	1728	$-1·06461 \pm 2·03720i$	1357
46	2·14651	1703	$-1·07326 \pm 2·05077i$	1337
47	2·16354	1678	$-1·08177 \pm 2·06414i$	1320
48	2·18032	1653	$-1·09016 \pm 2·07734i$	1303
49	2·19685	1630	$-1·09843 \pm 2·09037i$	1286
50	2·21315	1608	$-1·10658 \pm 2·10323i$	1268
51	2·22923	1586	$-1·11461 \pm 2·11591i$	1254
52	2·24509	1565	$-1·12254 \pm 2·12845i$	1239
53	2·26074		$-1·13037 \pm 2·14084i$	

TABLE 2 (contd.)

ROOTS OF EQUATION $4z^3+3z = \Delta$

Δ	z_1		z_2, z_3	
53	2·26074	1544	$-1·13037 \pm 2·14084i$	1224
54	2·27618	1525	$-1·13809 \pm 2·15308i$	1209
55	2·29143	1505	$-1·14571 \pm 2·16517i$	1196
56	2·30648	1487	$-1·15324 \pm 2·17713i$	1183
57	2·32135	1469	$-1·16068 \pm 2·18896i$	1168
58	2·33604	1452	$-1·16802 \pm 2·20064i$	1156
59	2·35056	1434	$-1·17528 \pm 2·21220i$	1144
60	2·36490	1418	$-1·18245 \pm 2·22364i$	1136
61	2·37908	1402	$-1·18954 \pm 2·23500i$	1115
62	2·39310	1387	$-1·19655 \pm 2·24615i$	1108
63	2·40697	1371	$-1·20348 \pm 2·25723i$	1098
64	2·42068	1357	$-1·21034 \pm 2·26821i$	1086
65	2·43425	1342	$-1·21712 \pm 2·27907i$	1077
66	2·44767	1329	$-1·22389 \pm 2·28984i$	1065
67	2·46096	1314	$-1·23048 \pm 2·30049i$	1054
68	2·47410	1302	$-1·23705 \pm 2·31103i$	1046
69	2·48712	1288	$-1·24356 \pm 2·32149i$	1035
70	2·50000	1276	$-1·25000 \pm 2·33184i$	1027
71	2·51276	1263	$-1·25638 \pm 2·34211i$	1017
72	2·52539	1252	$-1·26270 \pm 2·35228i$	1007
73	2·53791	1239	$-1·26895 \pm 2·36235i$	1000
74	2·55030	1228	$-1·27515 \pm 2·37235i$	990
75	2·56258	1217	$-1·28129 \pm 2·38225i$	983
76	2·57475	1206	$-1·28738 \pm 2·39208i$	974
77	2·58681	1195	$-1·29341 \pm 2·40182i$	965
78	2·59876	1185	$-1·29938 \pm 2·41147i$	957
79	2·61061	1174	$-1·30530 \pm 2·42104i$	951
80	2·62235	1165	$-1·31118 \pm 2·43055i$	942
81	2·63400	1154	$-1·31700 \pm 2·43997i$	935
82	2·64554	1145	$-1·32277 \pm 2·44932i$	927
83	2·65699	1135	$-1·32849 \pm 2·45859i$	921
84	2·66834	1126	$-1·33417 \pm 2·46780i$	913
85	2·67960	1117	$-1·33980 \pm 2·47693i$	907
86	2·69077	1109	$-1·34539 \pm 2·48600i$	900
87	2·70186	1099	$-1·35093 \pm 2·49500i$	892
88	2·71285	1091	$-1·35642 \pm 2·50392i$	888
89	2·72376	1082	$-1·36188 \pm 2·51280i$	880
90	2·73458	1075	$-1·36729 \pm 2·52160i$	873
91	2·74533	1066	$-1·37266 \pm 2·53033i$	868
92	2·75599		$-1·37799 \pm 2·53901i$	

TABLE 2 (contd.)

ROOTS OF EQUATION $4z^3+3z = \Delta$

Δ	z_1		z_2, z_3	
92	2·75599	1058	−1·37799±2·53901i	862
93	2·76657	1050	−1·38328±2·54763i	856
94	2·77707	1043	−1·38854±2·55619i	850
95	2·78750	1036	−1·39375±2·56469i	845
96	2·79786	1028	−1·39893±2·57314i	838
97	2·80814	1020	−1·40407±2·58152i	832
98	2·81834	1014	−1·40917±2·58984i	828
99	2·82848	1006	−1·41424±2·59812i	821
100	2·83854	1000	−1·41927±2·60633i	817
101	2·84854	993	−1·42427±2·61450i	813
102	2·85847	986	−1·42924±2·62263i	806
103	2·86833	980	−1·43417±2·63069i	801
104	2·87813	973	−1·43907±2·63870i	796
105	2·88786	967	−1·44393±2·64666i	792
106	2·89753	961	−1·44877±2·65458i	786
107	2·90714	955	−1·45357±2·66244i	782
108	2·91669	949	−1·45834±2·67026i	778
109	2·92618	942	−1·46309±2·67804i	772
110	2·93560	937	−1·46780±2·68576i	769
111	2·94497	931	−1·47248±2·69345i	763
112	2·95428	925	−1·47714±2·70108i	760
113	2·96353	920	−1·48177±2·70868i	754
114	2·97273	915	−1·48637±2·71622i	751
115	2·98188	908	−1·49094±2·72373i	746
116	2·99096	904	−1·49548±2·73119i	742
117	3·00000	898	−1·50000±2·73861i	738
118	3·00898	893	−1·50449±2·74599i	734
119	3·01791	888	−1·50896±2·75333i	731
120	3·02679	883	−1·51340±2·76064i	726
121	3·03562	878	−1·51781±2·76790i	722
122	3·04440	873	−1·52220±2·77512i	719
123	3·05313	869	−1·52657±2·78231i	714
124	3·06182	863	−1·53091±2·78945i	710
125	3·07045	859	−1·53522±2·79655i	708
126	3·07904	854	−1·53952±2·80363i	704
127	3·08758	849	−1·54379±2·81067i	700
128	3·09607	845	−1·54804±2·81767i	696
129	3·10452	841	−1·55226±2·82463i	693
130	3·11293	836	−1·55646±2·83156i	690
131	3·12129		−1·56065±2·83846i	

TABLE 2 (concluded)

ROOTS OF EQUATION $4z^3+3z = \Delta$

Δ	z_1		z_2, z_3	
131	3·12129	832	−1·56065±2·83846i	685
132	3·12961	827	−1·56480±2·84531i	683
133	3·13788	824	−1·56894±2·85214i	680
134	3·14612	819	−1·57306±2·85894i	676
135	3·15431	815	−1·57715±2·86570i	673
136	3·16246	811	−1·58123±2·87243i	669
137	3·17057	806	−1·58528±2·87912i	668
138	3·17863	803	−1·58932±2·88580i	662
139	3·18666	799	−1·59333±2·89242i	661
140	3·19465	795	−1·59733±2·89903i	656
141	3·20260	792	−1·60130±2·90559i	655
142	3.21052	787	−1·60526±2·91214i	652
143	3·21839	784	−1·60920±2·91866i	647
144	3.22623	780	−1·61311±2·92513i	645
145	3·23403	776	−1·61751±2·93158i	644
146	3·24179	773	−1·62090±2·93802i	639
147	3·24952	769	−1·62476±2·94441i	636
148	3·25721	766	−1·62860±2·95077i	634
149	3·26487	762	−1·63243±2·95711i	631
150	3·27249	758	−1·63624±2·96342i	629
151	3·28007	756	−1·64004±2·96971i	625
152	3·28763	752	−1·64386±2·97596i	623
153	3·29515	748	−1·64757±2·98219i	621
154	3·30263	745	−1·65132±2·98840i	617
155	3·31008	742	−1·65504±2·99457i	616
156	3·31750	739	−1·65875±3·00073i	613
157	3·32489	736	−1·66250±3·00686i	610
158	3·33225	732	−1·66612±3·01296i	608
159	3·33957	729	−1·66979±3·01904i	604
160	3·34686	726	−1·67343±3·02508i	602
161	3·35412	724	−1·67706±3·03110i	601
162	3·36136	720	−1·68068±3·03711i	598
163	3·36856	717	−1·68428±3·04309i	595
164	3.37573	714	−1·68786±3·04904i	593
165	3·38287	711	−1·69143±3·05497i	591
166	3·38998	708	−1·69499±3·06088i	588
167	3·39706	706	−1·69853±3·06676i	587
168	3·40412	702	−1·70206±3·07263i	583
169	3·41114	700	−1·70557±3·07846i	582
170	3·41814		−1·70907±3·08428i	

This book introduces a rapid and efficient method of finding the roots of an arbitrary cubic equation with real coefficients, by using specially computed 5-figure tables, and also a method of factorizing an arbitrary quartic equation by an appropriate use of a resolvent cubic. In each case, the solution is reduced to the few simplest possible operations, so that the roots of a cubic equation can be found with little more effort than those of a quadratic; the procedure for a quartic is also extremely easy. Special cases when the solution is particularly simple (e.g. those of multiple roots), and also the cases of "ill-conditioned" equations, are treated in detail.

A new approach is adopted which consists of a systematic and exhaustive treatment of the centuries-old basic theory, so as to enable the user to solve every cubic and quartic equation without any preliminary algebraic transformation, by a simple computation and use of the most convenient numerical tables.

A few examples illustrate both methods. The book can be understood by everybody who knows elementary algebra.

NEUMARK

SOLUTION OF CUBIC AND QUARTIC EQUATIONS

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